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Analytical Modeling of Vibration of Micropolar Plates

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Abstract

This paper presents an extension of mathematical static model to dynamic problems of micropolar elastic plates, recently developed by the authors. The dynamic model is based on the generalization of Hellinger-Prange-Reissner (HPR) variational principle for the linearized micropolar (Cosserat) elastodynamics. The vibration model incorporates high accuracy assumptions of the micropolar plate deformation. The computations predict additional natural frequencies, related with the material microstructure. These results are consistent with the size-effect principle known from the micropolar plate deformation. The classic Mindlin-Reissner plate resonance frequencies appear as a limiting case for homogeneous materials with no microstructure.

Keywords

Cosserat Materials, Plate Vibration, Frequencies of Transverse Micro-Vibration, Variational Principle

1. Introduction

Classical theory of elasticity ignores the size effects of the particles and their mutual rotational interactions, thus considering the material particles to have only three degrees of freedom that represent their macrodisplacements. The stress tensor is symmetric and the surface loads are assumed to be solely determined by the force vector. Classical theory of elasticity is widely used in engineering and is successfully applied under small deformations to such linear elastic materials as stainless steel, concrete, plastic, aluminium, etc. Many modern engineering materials, however, contain fibers, grains, pores or macromolecules, which in turn make them exhibit the deformation that cannot be adequately described by the classical elasticity (see, for example, the studies of a low-density polymeric foam in [1] [2]). The microstructure of the body also has an impact on the elastic vibrations

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with high frequencies and short wavelengths. The dynamic problems describe the appearance of the new types of waves that are not predicted by the classical elasticity [3]. Micropolar theory of elasticity describes the deformation of the materials with internal microstructure. The material particles have six degrees of freedom (macrodisplacements and microrotations) and the surface loads are assured by the force and moment vectors. This assumption leads to the introduction of the couple stress tensor and the asymmetry of the stress tensor. The examples of the materials that consider micropolar and exhibit nonclassical behavior include platelet and particulate composites, sandwich and grid structures, honeycombs, concrete with sand, ferroelectric and phononic crystals, polyfoams, and human bones [1] [4]-[12].

The first theory of elasticity that took into account the microstructure of the material was developed in 1909 by Cosserat brothers. They presented the equations of local balance of momenta for stress and couple stress, and the expressions for surface tractions and couples [13]. Many significant contributions were made by Eringen, who developed micromorphic and micropolar theories of solids, fluids, memory-dependent media, micro stretch solids and fluids and solved several problems in these fields [14]. In 1967, Eringen introduced a theory of plates in the framework of micropolar elasticity [15]. Eringen's theory assumes constant transverse variation of micropolar rotations and is based on a direct integration of the Cosserat elasticity equations. The micropolar plate theory based on the Reissner plate theory [16]-[18] is developed by the authors in [19]-[21]. The results of the preliminary computations for the micropolar plate theory based on the Reissner plate are compatible with the precision of the Reissner plate theory [22].

In this paper, we present an extension of our static approach to the dynamics of micropolar elastic plates. We reformulate a generalization of Hellinger-Prange-Reissner (HPR) variational principle [23] for elastodynamics of micropolar materials, and then, using our assumptions, we postulate the variational principle for the Cosserat plate dynamics. This principle allows us to obtain dynamic equilibrium equations and constitutive relations. We present our preliminary study of the influence of plate size effect on the natural frequencies in comparison with Mindlin-Reissner plates and perform the computations for different levels of the asymmetric microstructure.

2. Micropolar (Cosserat) Linear Elastodynamics

2.1. Fundamental Equations

Throughout this paper we will use the Einstein summation notation. The Latin subindices take values in the set $\{1,2,3\}$, while the Greek letters take the values 1 or 2.

The Cosserat linear elasticity balance equations without body forces represent the balance of linear and angular momentums of micropolar elastodynamics and have the following form:

$$\operatorname{div}\boldsymbol{\sigma} = \frac{\partial \boldsymbol{p}}{\partial t}, \tag{1}$$

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma} + \operatorname{div}\boldsymbol{\mu} = \frac{\partial \boldsymbol{q}}{\partial t}, \tag{2}$$

where the quantity $\boldsymbol{\sigma} = \sigma_{ji}$ is the stress tensor, $\boldsymbol{\mu} = \mu_{ji}$ the couple stress tensor, $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ the are displacement and rotation vectors, $\boldsymbol{p} = \rho\frac{\partial \boldsymbol{u}}{\partial t}$ and $\boldsymbol{q} = \boldsymbol{J}\frac{\partial \boldsymbol{\varphi}}{\partial t}$ are the linear and angular momenta, ρ and $\boldsymbol{J}$ are the material density and the rotatory inertia characteristics, $\boldsymbol{\varepsilon} = \varepsilon_{ijk}$ is the Levi-Civita tensor. In the linearized theory ρ and $\boldsymbol{J}$ are assumed to be constant [3] [14]. For simplicity we consider the case $\boldsymbol{J} = J\boldsymbol{I}$.

The linearized constitutive equations are given in the form [3]:

$$\boldsymbol{\sigma} = \boldsymbol{C}_\sigma[\gamma] = (\mu+\alpha)\boldsymbol{\gamma} + (\mu-\alpha)\boldsymbol{\gamma}^{\mathrm{T}} + \lambda(\operatorname{tr}\boldsymbol{\gamma})\mathbf{1}, \tag{3}$$

$$\boldsymbol{\mu} = \boldsymbol{C}_\mu[\chi] = (\gamma+\epsilon)\boldsymbol{\chi} + (\gamma-\epsilon)\boldsymbol{\chi}^{\mathrm{T}} + \beta(\operatorname{tr}\boldsymbol{\chi})\mathbf{1}, \tag{4}$$

and the strain-displacement and torsion-rotation relations

$$\boldsymbol{\gamma} = (\nabla\boldsymbol{u})^{\mathrm{T}} + \boldsymbol{\varepsilon}\cdot\boldsymbol{\varphi} \quad \text{and} \quad \boldsymbol{\chi} = (\nabla\boldsymbol{\varphi})^{\mathrm{T}}, \tag{5}$$

where μ, λ are the symmetric and α, β, γ, ϵ the asymmetric Cosserat elasticity constants.

The constitutive equations in the reverse form can be written as

$$\gamma = \boldsymbol{K}_\gamma[\boldsymbol{\sigma}] = (\mu' + \alpha')\boldsymbol{\sigma} + (\mu' - \alpha')\boldsymbol{\sigma}^{\mathrm{T}} + \lambda'(\mathrm{tr}\boldsymbol{\sigma})\mathbf{1}, \tag{6}$$

$$\boldsymbol{\chi} = \boldsymbol{K}_\chi[\boldsymbol{\mu}] = (\gamma' + \epsilon')\boldsymbol{\mu} + (\gamma' - \epsilon')\boldsymbol{\mu}^{\mathrm{T}} + \beta'(\mathrm{tr}\boldsymbol{\mu})\mathbf{1}. \tag{7}$$

where $\mu' = \frac{1}{4\mu}$, $\alpha' = \frac{1}{4\alpha}$, $\gamma' = \frac{1}{4\gamma}$, $\epsilon' = \frac{1}{4\epsilon}$, $\lambda' = \frac{-\lambda}{6\mu\left(\lambda + \frac{2\mu}{3}\right)}$ and $\beta' = \frac{-\beta}{6\mu\left(\beta + \frac{2\gamma}{3}\right)}$.

We consider a Cosserat elastic body B_0. The equilibrium Equations (1) and (2) with constitutive formulas (3)-(4) and kinematics formulas (5) should be accompanied by the following mixed boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}_0, \boldsymbol{\varphi} = \boldsymbol{\varphi}_0 \text{ on } \mathcal{G}_1^t = \partial B_0 \setminus \partial B_\sigma \times [t_0, t], \tag{8}$$

$$\boldsymbol{\sigma}_{\boldsymbol{n}} = \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_0, \ \boldsymbol{\mu}_{\boldsymbol{n}} = \boldsymbol{\mu} \cdot \boldsymbol{n} = \boldsymbol{\mu}_0 \text{ on } \mathcal{G}_2^t = \partial B_\sigma \times [t_0, t], \tag{9}$$

and initial conditions

$$\boldsymbol{u}(x,0) = \boldsymbol{U}_0, \ \boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{0}) = \boldsymbol{\Phi}_0 \text{ in } B_0, \tag{10}$$

$$\dot{\boldsymbol{u}}(x,0) = \bar{\boldsymbol{U}}_0, \ \dot{\boldsymbol{\varphi}}(\boldsymbol{x},\boldsymbol{0}) = \bar{\boldsymbol{\Phi}}_0 \text{ in } B_0, \tag{11}$$

where $\boldsymbol{u}_0$ and $\boldsymbol{\varphi}_0$ are prescribed on $\mathcal{G}_1$, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\mu}_0$ on $\mathcal{G}_2$, and $\boldsymbol{n}$ denotes the outward unit normal vector to ∂B_0.

2.2. Cosserat Elastic Energy

The strain stored energy U_C of the body B_0 is defined by the integral [3]:

$$U_C = \int_{B_0} W\{\boldsymbol{\gamma}, \boldsymbol{\chi}\} \mathrm{d}v, \tag{12}$$

where non-negative

$$\begin{aligned} W\{\boldsymbol{\gamma}, \boldsymbol{\chi}\} = &\frac{\mu + \alpha}{2}\gamma_{ij}\gamma_{ij} + \frac{\mu - \alpha}{2}\gamma_{ij}\gamma_{ji} + \frac{\lambda}{2}\gamma_{kk}\gamma_{nn} \\ &+ \frac{\gamma + \varepsilon}{2}\chi_{ij}\chi_{ij} + \frac{\gamma - \varepsilon}{2}\chi_{ij}\chi_{ji} + \frac{\beta}{2}\chi_{kk}\chi_{nn}, \end{aligned} \tag{13}$$

then the constitutive relations (3)-(4) can be written in the form:

$$\boldsymbol{\sigma} = \boldsymbol{C}_\sigma[\boldsymbol{\gamma}] = \nabla_\gamma W \text{ and } \boldsymbol{\mu} = \boldsymbol{C}_\mu[\boldsymbol{\chi}] = \nabla_\chi W. \tag{14}$$

For future convenience, we present the stress energy

$$U_K = \int_{B_0} \Phi\{\boldsymbol{\sigma}, \boldsymbol{\mu}\} \mathrm{d}v,$$

where

$$\begin{aligned} \Phi\{\boldsymbol{\sigma}, \boldsymbol{\mu}\} = &\frac{\mu' + \alpha'}{2}\sigma_{ij}\sigma_{ij} + \frac{\mu' - \alpha'}{2}\sigma_{ij}\sigma_{ji} + \frac{\lambda'}{2}\sigma_{kk}\sigma_{nn} \\ &+ \frac{\gamma' + \epsilon'}{2}\mu_{ij}\mu_{ij} + \frac{\gamma' - \epsilon'}{2}\mu_{ij}\mu_{ji} + \frac{\beta'}{2}\mu_{kk}\mu_{nn}. \end{aligned} \tag{15}$$

The constitutive relations in the reverse form (6)-(7) can be also written in form:

$$\boldsymbol{\gamma} = \boldsymbol{K}_\gamma[\boldsymbol{\sigma}] = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}, \boldsymbol{\chi} = \boldsymbol{K}_\chi[\boldsymbol{\mu}] = \frac{\partial \Phi}{\partial \boldsymbol{\mu}}. \tag{16}$$

The total internal work done by the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ over the strains $\boldsymbol{\gamma}$ and $\boldsymbol{\chi}$ for the body B_0 [3] is

$$U = \int_{B_0} [\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} + \boldsymbol{\mu} \cdot \boldsymbol{\chi}] \mathrm{d}v \tag{17}$$

and

$$U = U_K = U_C$$

provided the constitutive relations (3)-(4) hold.

We also consider the stored kinetic energy of the body B_0 defined by the integral

$$T_C = \int_{B_0} \Upsilon_C \left\{ \frac{\partial \boldsymbol{u}}{\partial t}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right\} \mathrm{d}v = \frac{1}{2} \int_{B_0} \left(\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} \right)^2 + J \left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \right)^2 \right) \mathrm{d}v,$$

We also present the kinetic energy as

$$T_K = \int_{B_0} \Upsilon_K \{\boldsymbol{p}, \boldsymbol{q}\} \mathrm{d}v = \frac{1}{2} \int_{B_0} \left(\frac{\boldsymbol{p}^2}{\rho} + \frac{\boldsymbol{q}^2}{J} \right) \mathrm{d}v,$$

where

$$\boldsymbol{p} = \frac{\partial \Upsilon_C}{\partial \dot{\boldsymbol{u}}} = \boldsymbol{K}_p \left[\frac{\partial \boldsymbol{u}}{\partial t} \right] = \rho \frac{\partial \boldsymbol{u}}{\partial t} \text{ and } \boldsymbol{q} = \frac{\partial \Upsilon_C}{\partial \dot{\boldsymbol{\varphi}}} = \boldsymbol{K}_{\dot{\varphi}} \left[\frac{\partial \boldsymbol{\varphi}}{\partial t} \right] = J \frac{\partial \boldsymbol{\varphi}}{\partial t}, \tag{18}$$

or

$$T_I = \int_{B_0} \Upsilon_I \mathrm{d}v = \frac{1}{2} \int_{B_0} \left(\frac{\partial \boldsymbol{u}}{\partial t} \boldsymbol{p} + \frac{\partial \boldsymbol{\varphi}}{\partial t} \boldsymbol{q} \right) \mathrm{d}v$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \Upsilon_K}{\partial \boldsymbol{p}} = \boldsymbol{K}_{\dot{u}} [\boldsymbol{p}] = \frac{\boldsymbol{p}}{\rho} \text{ and } \frac{\partial \boldsymbol{\varphi}}{\partial t} = \frac{\partial \Upsilon_K}{\partial \boldsymbol{q}} = \boldsymbol{K}_{\dot{\varphi}} [\boldsymbol{q}] = \frac{\boldsymbol{q}}{J}, \tag{19}$$

The internal work done by the inertia forces over displacement and microrotation is

$$T_W = \int_{B_0} \Upsilon_W \mathrm{d}v = \int_{B_0} \left(\frac{\partial \boldsymbol{p}}{\partial t} \cdot \boldsymbol{u} + \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{t}} \cdot \boldsymbol{\varphi} \right) \mathrm{d}v$$

Using the integration by parts

$$\int_{t_0}^{t_k} T_K \mathrm{d}t = \int_{t_0}^{t_k} T_I \mathrm{d}t = \frac{1}{2} \int_{B_0} (\boldsymbol{p} \cdot \boldsymbol{u} + \boldsymbol{q} \cdot \boldsymbol{\varphi}) \mathrm{d}v \Big|_{t_0}^{t_k} - \int_{t_0}^{t_k} T_W \mathrm{d}t$$

and taking into account the zero variation of $\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{q}$ and $\boldsymbol{\varphi}$ at the end points of the time interval $[t_0, t_k]$ we obtain

$$\delta \int_{t_0}^{t_k} T_K = -\delta \int_{t_0}^{t_k} T_W \quad \text{or} \quad \delta T_C = \delta T_K = -\delta T_W$$

Note that since the variations $\delta \boldsymbol{u}$ and $\delta \boldsymbol{\varphi}$ at and points t_0 and t_k are zeros then

$$\int_{t_0}^{t_k} \int_{B_0} \left(\left(\boldsymbol{p} \cdot \delta \left(\frac{\partial \boldsymbol{u}}{\partial t} \right) + \boldsymbol{q} \cdot \delta \left(\frac{\partial \boldsymbol{\varphi}}{\partial t} \right) \right) \right) \mathrm{d}v\mathrm{d}t = -\int_{t_0}^{t_k} \int_{B_0} \left(\frac{\partial \boldsymbol{p}}{\partial t} \cdot \delta \boldsymbol{u} + \frac{\partial \boldsymbol{q}}{\partial t} \cdot \delta \boldsymbol{\varphi} \right) \mathrm{d}v\mathrm{d}t \tag{20}$$

2.3. Hellinger-Prange-Reissner (HPR) Principle for Elastodynamics

We modify the HPR principle [23] for the case of Cosserat elastodynamics in the following way. Now it states, that for any set $\mathcal{A}$ of all admissible states $\mathfrak{s} = [\boldsymbol{u}, \boldsymbol{\varphi}, \boldsymbol{\gamma}, \boldsymbol{\chi}, \boldsymbol{\sigma}, \boldsymbol{\mu}]$ that satisfy the strain-displacement and torsion-rotation relations (5), the zero variation

$$\delta \Theta(\mathfrak{s}) = 0$$

of the functional

$$\begin{aligned} \Theta(\mathfrak{s}) = & \int_{t_0}^{t_k} \left[U_K + T_C - \int_{B_0} \left(\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} + \boldsymbol{\mu} \cdot \boldsymbol{\chi} + \boldsymbol{p} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{q} \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) \mathrm{d}v \right] \mathrm{d}t \\ & + \int_{t_0}^{t} \int_{\mathcal{G}_1} \left[\boldsymbol{\sigma}_{\boldsymbol{n}} \cdot (\boldsymbol{u} - \boldsymbol{u}_0) + \boldsymbol{\mu}_{\boldsymbol{n}} (\boldsymbol{\varphi} - \boldsymbol{\varphi}_0) \right] \mathrm{d}a\mathrm{d}t + \int_{t_0}^{t} \int_{\mathcal{G}_2} \left[\boldsymbol{\sigma}_0 \cdot \boldsymbol{u} + \boldsymbol{\mu}_0 \cdot \boldsymbol{\varphi} \right] \mathrm{d}a\mathrm{d}t \end{aligned} \tag{21}$$

at $\mathfrak{s} \in \mathcal{A}$ is equivalent of $\mathfrak{s}$ to be a solution of the system of equilibrium Equations (1) and (2), constitutive relations (6)-(7), which satisfies the mixed boundary conditions (8)-(9).

Proof of the Principle

Let us consider the variation of the functional $\Theta(\mathfrak{s})$:

$$\delta\Theta(\mathfrak{s}) = \int_{t_0}^{t_k}\left[\delta U_K + \delta T_C\right]\mathrm{d}t - \int_{t_0}^{t_k}\int_{B_0}\delta\boldsymbol{\sigma}\cdot\boldsymbol{\gamma} + \boldsymbol{\sigma}\cdot\delta\boldsymbol{\gamma} + \delta\boldsymbol{\mu}\cdot\boldsymbol{\chi} + \boldsymbol{\mu}\cdot\delta\boldsymbol{\chi}$$
$$+\frac{\partial \boldsymbol{u}}{\partial t}\delta\boldsymbol{p} + \boldsymbol{p}\cdot\delta\left(\frac{\partial \boldsymbol{u}}{\partial t}\right) + \delta\boldsymbol{q}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t} + \boldsymbol{q}\cdot\delta\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}\right)\mathrm{d}v\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{\mathcal{G}_1}\left[\delta\boldsymbol{\sigma}_{\boldsymbol{n}}\cdot(\boldsymbol{u}-\boldsymbol{u}_0) + \boldsymbol{\sigma}_{\boldsymbol{n}}\delta\boldsymbol{u} + \delta\boldsymbol{\mu}_{\boldsymbol{n}}\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_0) + \boldsymbol{\mu}_{\boldsymbol{n}}\delta\boldsymbol{\varphi}\right]\mathrm{d}a\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{\mathcal{G}_2}\left[\boldsymbol{\sigma}_0\cdot\delta\boldsymbol{u} + \boldsymbol{\mu}_0\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}a\mathrm{d}t$$

Taking into account (5), we can perform the integration by parts

$$\int_{B_0}\boldsymbol{\sigma}\cdot\delta\boldsymbol{\gamma}\mathrm{d}v = \int_{\partial B_0}\boldsymbol{\sigma}_{\boldsymbol{n}}\cdot\delta\boldsymbol{u}\mathrm{d}a - \int_{B_0}\delta\boldsymbol{u}\cdot\mathrm{div}\boldsymbol{\sigma}\mathrm{d}v + \int_{B_0}\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\delta\boldsymbol{\varphi}\mathrm{d}v$$
$$\int_{B_0}\boldsymbol{\mu}\cdot\delta\boldsymbol{\chi}\mathrm{d}v = \int_{\partial B_0}\boldsymbol{\mu}_{\boldsymbol{n}}\cdot\delta\boldsymbol{\varphi}\mathrm{d}a - \int_{B_0}\delta\boldsymbol{\varphi}\cdot\mathrm{div}\boldsymbol{\mu}\mathrm{d}v$$

and based on (16)-(19)

$$\delta\Phi = \frac{\partial\Phi}{\partial\boldsymbol{\sigma}}\cdot\delta\boldsymbol{\sigma} + \frac{\partial\Phi}{\partial\boldsymbol{\mu}}\cdot\delta\boldsymbol{\mu} = \boldsymbol{K}_{\gamma}[\boldsymbol{\sigma}]\cdot\delta\boldsymbol{\sigma} + \boldsymbol{K}_{\chi}[\boldsymbol{\mu}]\cdot\delta\boldsymbol{\mu},$$
$$\delta\Upsilon_C = \frac{\partial\Upsilon_C}{\partial\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)}\cdot\delta\left(\frac{\partial\boldsymbol{u}}{\partial t}\right) + \frac{\partial\Upsilon_C}{\partial\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}\right)}\cdot\delta\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}\right) = \boldsymbol{K}_{\boldsymbol{p}}\left[\frac{\partial\boldsymbol{u}}{\partial t}\right]\cdot\delta\left(\frac{\partial\boldsymbol{u}}{\partial t}\right) + \boldsymbol{K}_{\dot{\varphi}}\left[\frac{\partial\boldsymbol{\varphi}}{\partial t}\right]\cdot\delta\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}\right).$$

Then, keeping in mind $\delta T_K = -\delta T$ and (20), we can rewrite the expression for the variation of the functional $\delta\Theta(\mathfrak{s})$ in the following form

$$\delta\Theta(\mathfrak{s}) = \int_{t_0}^{t}\int_{B_0}\left[\left(\boldsymbol{K}_{\gamma}^{-1}[\boldsymbol{\sigma}]-\boldsymbol{\gamma}\right)\cdot\delta\boldsymbol{\sigma}\right]\mathrm{d}v\mathrm{d}t + \int_{t_0}^{t}\int_{B_0}\left[\left(\boldsymbol{K}_{\chi}^{-1}[\boldsymbol{\mu}]-\boldsymbol{\chi}\right)\cdot\delta\boldsymbol{\mu}\right]\mathrm{d}v\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{B_0}\left[\left(\boldsymbol{K}_{\boldsymbol{p}}\left[\frac{\partial\boldsymbol{u}}{\partial t}\right]-\boldsymbol{p}\right)\cdot\delta\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)\right]\mathrm{d}v\mathrm{d}t + \int_{t_0}^{t}\int_{B_0}\left[\left(\boldsymbol{K}_{\dot{\varphi}}\left[\frac{\partial\boldsymbol{\varphi}}{\partial t}\right]-\boldsymbol{q}\right)\cdot\delta\left(\frac{\partial\boldsymbol{\varphi}}{\partial t}\right)\right]\mathrm{d}v\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{B_0}\left[\left(\mathrm{div}\boldsymbol{\sigma}-\frac{\partial\boldsymbol{p}}{\partial t}\right)\cdot\delta\boldsymbol{u}\right]\mathrm{d}v\mathrm{d}t + \int_{t_0}^{t}\int_{B_0}\left[\left(\mathrm{div}\boldsymbol{\mu}+\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}-\frac{\partial\boldsymbol{q}}{\partial t}\right)\cdot\delta\boldsymbol{u}\right]\mathrm{d}v\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{\mathcal{G}_1}\left[(\boldsymbol{u}-\boldsymbol{u}_0)\cdot\delta\boldsymbol{\sigma}_{\boldsymbol{n}}\right]\mathrm{d}a\mathrm{d}t + \int_{t_0}^{t}\int_{\mathcal{G}_1}\left[(\boldsymbol{\varphi}-\boldsymbol{\varphi}_0)\cdot\delta\boldsymbol{\mu}_{\boldsymbol{n}}\right]\mathrm{d}a\mathrm{d}t$$
$$+\int_{t_0}^{t}\int_{\mathcal{G}_2}\left[(\boldsymbol{\sigma}_{\boldsymbol{n}}-\boldsymbol{\sigma}_0)\cdot\delta\boldsymbol{u}\right]\mathrm{d}a\mathrm{d}t + \int_{t_0}^{t}\int_{\mathcal{G}_2}\left[(\boldsymbol{\mu}_{\boldsymbol{n}}-\boldsymbol{\mu}_0)\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}a\mathrm{d}t$$

The latter expression provides the proof of the principle.

3. Review of Cosserat Plate Assumptions

In this section we review our stress, couple stress and kinematic assumptions of the Cosserat plate [20]. We consider the thin plate P, where h is the thickness of the plate and $x_3 = 0$ contains its middle plane. The sets T and B are the top and bottom surfaces contained in the planes $x_3 = h/2$, $x_3 = -h/2$ respectively and the curve Γ is the boundary of the middle plane of the plate.

The set of points $P = \left(\Gamma\times\left[-\frac{h}{2},\frac{h}{2}\right]\right)\cup T\cup B$ forms the entire surface of the plate and $\Gamma_u\times\left[-\frac{h}{2},\frac{h}{2}\right]$ is the lateral part of the boundary where displacements and microrotations are prescribed. The notation $\Gamma_\sigma = \Gamma\setminus\Gamma_u$

of the remainder we use to describe the lateral part of the boundary edge $\Gamma_\sigma \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ where stress and couple stress are prescribed. We also use notation P_0 for the middle plane internal domain of the plate.

In our case we consider the vertical load and pure twisting momentum boundary conditions at the top and bottom of the plate, which can be written in the form:

$$\sigma_{33}(x_1, x_2, h/2) = \sigma^t(x_1, x_2, t), \ \sigma_{33}(x_1, x_2, -h/2) = \sigma^b(x_1, x_2, t), \tag{22}$$

$$\sigma_{3\beta}(x_1, x_2, \pm h/2) = 0, \tag{23}$$

$$\mu_{33}(x_1, x_2, h/2) = \mu^t(x_1, x_2, t), \ \mu_{33}(x_1, x_2, -h/2) = \mu^b(x_1, x_2, t), \tag{24}$$

$$\mu_{3\beta}(x_1, x_2, \pm h/2) = 0, \tag{25}$$

where $(x_1, x_2) \in P_0$.

Some basic stress and kinematic assumptions are similar to the Reissner plate theory [16] and other chosen to be consistent with the micropolar elasticity equilibrium equations.

3.1. Stress and Couple Stress Assumptions

We reproduce the main micropolar plate assumptions presented in [20]. The variation of stress σ_{ij} and couple stress μ_{ij} components across the thickness is represented by means of polynomials of x_3.

$$\sigma_{\alpha\beta} = \frac{h}{2}\zeta m_{\alpha\beta}(x_1, x_2, t), \tag{26}$$

$$\sigma_{3\beta} = q_\beta(x_1, x_2, t)(1-\zeta^2), \tag{27}$$

$$\sigma_{\beta 3} = q_\beta^*(x_1, x_2, t)(1-\zeta^2) + \hat{q}_\beta(x_1, x_2, t), \tag{28}$$

$$\sigma_{33} = -\frac{3}{4}\left(\frac{1}{3}\zeta^3 - \zeta\right)p_1(x_1, x_2, t) + \zeta p_2(x_1, x_2, t) + \sigma_0(x_1, x_2, t), \tag{29}$$

$$\mu_{\alpha\beta} = (1-\zeta^2) r_{\alpha\beta}(x_1, x_2, t) + r_{\alpha\beta}^*(x_1, x_2, t), \tag{30}$$

$$\mu_{\beta 3} = \frac{h}{2}\zeta s_\beta^*(x_1, x_2, t), \tag{31}$$

$$\mu_{3\beta} = \left(\frac{1}{3}\zeta^3 - \zeta\right)s_\beta(x_1, x_2, t), \tag{32}$$

$$\mu_{3\beta} = 0, \tag{33}$$

$$\mu_{33} = \zeta v + v^*, \tag{34}$$

where $p_1(x_1, x_2, t) = \eta p(x_1, x_2, t)$, $p_2(x_1, x_2, t) = \frac{(1-\eta)}{2} p(x_1, x_2, t), \eta \in [0,1]$ is the splitting parameter, the functions $v(x_1, x_2, t)$ and $v^*(x_1, x_2, t)$ are defined as $v(x_1, x_2, t) = \frac{1}{2}\left(\mu^t(x_1, x_2, t) - \mu^b(x_1, x_2, t)\right)$ and $v^*(x_1, x_2, t) = \frac{1}{2}\left(\mu^t(x_1, x_2, t) + \mu^b(x_1, x_2, t)\right)$, $\zeta = \frac{2}{h}x_3$, and $\alpha, \beta \in \{1, 2\}$. In the future consideration $v = 0$ and $v^* = 0$.

We also will use the notation of the normalized components of the micropolar plate stress set $\mathcal{S}^{(\eta)} = \left[M_{\alpha\beta}, Q_\alpha, Q_\alpha^*, \hat{Q}_\alpha, R_{\alpha\beta}, R_{\alpha\beta}^*, S_\beta^*\right]$ are defined as

$$M_{\alpha\beta} = \left(\frac{h}{2}\right)^2 \int_{-1}^{1} \zeta_3 \sigma_{\alpha\beta} \mathrm{d}\zeta_3 = \frac{h^3}{12} m_{\alpha\beta}, \tag{35}$$

$$Q_\alpha = \frac{h}{2}\int_{-1}^{1}\sigma_{3\alpha}\mathrm{d}\zeta_3 = \frac{2h}{3}q_\alpha, \tag{36}$$

$$Q_\alpha^* = \frac{h}{2}\int_{-1}^{1}q_\alpha^*\left(1-\zeta^2\right)\mathrm{d}\zeta_3 = \frac{2h}{3}q_\alpha^*, \tag{37}$$

$$\hat{Q}_\alpha = \frac{h}{2}\int_{-1}^{1}\hat{q}_\alpha\left(1-\zeta^2\right)\mathrm{d}\zeta_3 = \frac{2h}{3}\hat{q}_\alpha \tag{38}$$

$$R_{\alpha\beta} = \frac{h}{2}\int_{-1}^{1}r_{\alpha\beta}\left(1-\zeta^2\right)\mathrm{d}\zeta_3 = \frac{2h}{3}r_{\alpha\beta}, \tag{39}$$

$$R_{\alpha\beta}^* = \frac{h}{2}\int_{-1}^{1}r_{\alpha\beta}^*\left(1-\zeta^2\right)\mathrm{d}\zeta_3 = \frac{2h}{3}r_{\alpha\beta}^*, \tag{40}$$

$$S_\alpha^* = \left(\frac{h}{2}\right)^2\int_{-1}^{1}\zeta_3\mu_{\alpha3}\mathrm{d}\zeta_3 = \frac{h^3}{12}s_\alpha^*, \tag{41}$$

Here, M_{11} and M_{22} are the bending moments, M_{12} and M_{21}—the twisting moments, Q_α—the shear forces, Q_α^*, and $\hat{Q}_\alpha$—the transverse shear forces, $R_{11}, R_{22}, R_{11}^*, R_{22}^*$—the micropolar bending moments, $R_{12}, R_{21}, R_{12}^*, R_{21}^*$—the micropolar twisting moments, S_α^*—the micropolar couple moments, all defined per unit length.

3.2. Kinematics Assumptions

$$u_\alpha = \frac{h}{2}\zeta\Psi_\alpha\left(x_1, x_2, t\right), \tag{42}$$

$$u_3 = W\left(x_1, x_2, t\right) + \left(1-\zeta^2\right)W^*\left(x_1, x_2, t\right), \tag{43}$$

$$\varphi_\alpha = \frac{5}{4}\Omega_\alpha^0\left(x_1, x_2, t\right)\left(1-\zeta^2\right) + \hat{\Omega}_\alpha\left(x_1, x_2, t\right), \tag{44}$$

$$\varphi_3 = \zeta\Omega_3\left(x_1, x_2, t\right). \tag{45}$$

where

$$\Psi_\alpha = \frac{3}{h}\int_{-1}^{1}\zeta_3 u_\alpha \mathrm{d}\zeta_3,$$

$$W = \frac{3}{4}\int_{-1}^{1}\left(1-\zeta^2\right)u_3\mathrm{d}\zeta_3,$$

$$W^* = \frac{3}{4}\left(1-\zeta^2\right)^2 u_3\mathrm{d}\zeta_3,$$

$$\Omega_\alpha^0 = \frac{3}{4}\int_{-1}^{1}\left(1-\zeta^2\right)\varphi_\alpha\mathrm{d}\zeta_3,$$

$$\hat{\Omega}_\alpha = \frac{3}{4}\int_{-1}^{1}\varphi_\alpha\mathrm{d}\zeta_3,$$

$$\Omega_3 = \frac{3}{h}\int_{-1}^{1}\zeta_3\varphi_3\mathrm{d}\zeta_3, \tag{46}$$

$$U_\alpha = \frac{1}{2}\int_{-1}^{1}u_\alpha\mathrm{d}\zeta_3,$$

$$\Omega_3^0 = \frac{1}{2}\int_{-1}^{1}\varphi_3\mathrm{d}\zeta_3.$$

The terms Ψ_α represent the rotations, $W+W^*$ vertical displacement, $\frac{5}{4}\Omega_\alpha^0+\hat{\Omega}_\alpha$ describe microrotation components, and Ω_3 the slope at the middle plane of the plate. Thus, the transverse variation effect of microrotations is not neglected in our kinematic assumptions.

The components of the corresponding micropolar plate strain set $\mathcal{E}^{(\eta)}=\left[e_{\alpha\beta},\omega_\beta,\omega_\alpha^*,\hat{\omega}_\alpha,\tau_{3\alpha},\tau_{\alpha\beta},\tau_{\alpha\beta}^*\right]$ are defined as

$$e_{\alpha\beta}=\frac{3}{h}\int_{-1}^{1}\zeta_3\gamma_{\alpha\beta}\mathrm{d}\zeta_3, \tag{47}$$

$$\omega_\alpha=\frac{3}{4}\int_{-1}^{1}\gamma_{3\alpha}\left(1-\zeta^2\right)\mathrm{d}\zeta_3, \tag{48}$$

$$\omega_\alpha^*=\frac{3}{4}\int_{-1}^{1}\gamma_{\alpha3}\left(1-\zeta^2\right)\mathrm{d}\zeta_3, \tag{49}$$

$$\hat{\omega}_\alpha=\frac{3}{4}\int_{-1}^{1}\gamma_{\alpha3}\mathrm{d}\zeta_3, \tag{50}$$

$$\tau_{3\alpha}=\frac{3}{h}\int_{-1}^{1}\zeta_3\chi_{3\alpha}\mathrm{d}\zeta_3, \tag{51}$$

$$\tau_{\alpha\beta}=\frac{3}{4}\int_{-1}^{1}\chi_{\alpha\beta}\left(1-\zeta^2\right)\mathrm{d}\zeta_3, \tag{52}$$

$$\tau_{\alpha\beta}^*=\frac{3}{4}\int_{-1}^{1}\chi_{\alpha\beta}\mathrm{d}\zeta_3. \tag{53}$$

The components of Cosserat plate strain can also be represented in terms of the components of set $\mathcal{U}$ by the following formulas:

$$\begin{aligned}
e_{\alpha\beta}&=\Psi_{\beta,\alpha}+\varepsilon_{3\alpha\beta}\Omega_3,\\
\omega_\alpha&=\Psi_\alpha+\varepsilon_{3\alpha\beta}\left(\Omega_\beta^0+\hat{\Omega}_\beta\right),\\
\omega_\alpha^*&=W_{,\alpha}+\frac{4}{5}W_{,\alpha}^*+\varepsilon_{3\alpha\beta}\left(\Omega_\beta^0+\hat{\Omega}_\beta\right),\\
\hat{\omega}_\alpha&=\frac{3}{2}W_{,\alpha}+W_{,\alpha}^*+\varepsilon_{3\alpha\beta}\left(\frac{5}{4}\Omega_\beta^0+\frac{3}{2}\hat{\Omega}_\beta\right),\\
\tau_{3\alpha}&=\Omega_{3,\alpha},\\
\tau_{\alpha\beta}&=\Omega_{\beta,\alpha}^0+\hat{\Omega}_{\beta,\alpha},\\
\tau_{\alpha\beta}^*&=\frac{5}{4}\Omega_{\beta,\alpha}^0+\frac{3}{2}\hat{\Omega}_{\beta,\alpha},
\end{aligned} \tag{54}$$

The formulas (54) are called the Cosserat plate strain-displacement relation.

We also assume that the initial condition can be presented in the similar form:

$$\begin{gathered}
\Psi_\alpha\left(x_1,x_2,0\right)=\Psi_\alpha^0\left(x_1,x_2\right),\\
W\left(x_1,x_2,0\right)=w_0\left(x_1,x_2\right),\\
W^*\left(x_1,x_2,0\right)=w_0^*\left(x_1,x_2\right),\\
\Theta_\alpha^0\left(x_1,x_2,0\right)=\Theta_\alpha^0\left(x_1,x_2\right),\\
\hat{\Theta}_\alpha^0\left(x_1,x_2,0\right)=\hat{\Theta}_\alpha^0\left(x_1,x_2\right),
\end{gathered}$$

$$\Theta_3(x_1,x_2,0)=\Theta_3(x_1,x_2),$$

$$\frac{\partial\Psi_\alpha}{\partial t}(x_1,x_2,0)=\bar{\Psi}^0_\alpha(x_1,x_2),$$

$$\frac{\partial W}{\partial t}(x_1,x_2,0)=\bar{w}_0(x_1,x_2),$$

$$\frac{\partial W^*}{\partial t}(x_1,x_2,0)=\bar{w}^*_0(x_1,x_2),$$

$$\frac{\partial\hat{\Theta}^0_\alpha}{\partial t}(x_1,x_2,0)=\bar{\Theta}^0_\alpha(x_1,x_2),$$

$$\frac{\partial\hat{\Theta}^0_\alpha}{\partial t}(x_1,x_2,0)=\tilde{\Theta}^0_\alpha(x_1,x_2),$$

$$\frac{\partial\hat{\Theta}_3}{\partial t}(x_1,x_2,0)=\bar{\Theta}_3(x_1,x_2),$$

4. Specification of HPR Variational Principle for the Cosserat Plate Dynamics

The HPR variational principle for a Cosserat plate dynamics is most appropriately expressed in terms of corresponding integrands calculated across the whole thickness. We also introduce the weighted characteristics of displacements, microrotations, strains and stresses of the plate, which will be used to produce the explicit forms of these integrands.

4.1. The Cosserat Plate Elastic Stress Energy Density

We define the plate stress energy density by the formula [20];

$$\Phi(\mathcal{S})=\frac{h}{2}\int_{-1}^{1}\Phi\{\boldsymbol{\sigma},\boldsymbol{\mu}\}\,\mathrm{d}\zeta_3, \tag{55}$$

$$\begin{aligned}\Phi(\mathcal{S})=&-\frac{3\lambda(M_{\alpha\alpha})(M_{\beta\beta})}{h^3\mu(3\lambda+2\mu)}+\frac{3(\alpha+\mu)M^2_{\alpha\beta}}{2h^3\alpha\mu}+\frac{3(\alpha+\mu)}{160h^3\alpha\mu}\left[8\hat{Q}_\alpha\hat{Q}_\alpha+15Q_\alpha\hat{Q}_\alpha+20\hat{Q}_\alpha Q^*_\alpha+8Q^*_\alpha Q^*_\alpha\right]\\&+\frac{3(\alpha-\mu)M^2_{\alpha\beta}}{2h^3\alpha\mu}+\frac{\alpha-\mu}{280h^3\alpha\mu}\left[21Q_\alpha\left(5\hat{Q}_\alpha+4Q^*_\alpha\right)\right]-\frac{\gamma-\epsilon}{160h\gamma\epsilon}\left[24R^2_{\alpha\alpha}+45R^*_{\alpha\alpha}\right.\\&\left.+60R_{\alpha\beta}R^*_{\alpha\beta}+48R_{12}R_{21}\right]+\frac{3(\gamma+\epsilon)S^*_\alpha S^*_\alpha}{2h^3\gamma\epsilon}+\frac{\gamma+\epsilon}{160h^3\gamma\epsilon}\left[8R^2_{\alpha\beta}+15R^*_{\alpha\beta}R^*_{\alpha\beta}+20R_{\alpha\beta}R^*_{\alpha\beta}\right]\\&-\frac{3\beta}{80h\gamma(3\beta+2\gamma)}\left[-8(R_{\alpha\alpha})(R_{\beta\beta})-15(R^*_{\alpha\alpha})(R^*_{\beta\beta})-20(R_{\alpha\alpha})(R^*_{\alpha\alpha})\right]\\&-\frac{\beta}{4\gamma(3\beta+2\gamma)}\left[\left(2R_{\alpha\alpha}+3R^*_{\alpha\alpha}\right)t-h\left(v^2+t^2\right)\right]+\frac{\lambda}{560h\mu(3\lambda+2\mu)}\left[\frac{5+3\eta}{(1+\eta)}pM_{\alpha\alpha}\right]\\&+\frac{(\lambda+\mu)h}{840\mu(3\lambda+2\mu)}\left(\frac{140+168\eta+51\eta^2}{4(1+\eta)^2}\right)p^2+\frac{(\lambda+\mu)h}{2\mu(3\lambda+2\mu)}\sigma_0^2+\frac{\epsilon h}{12h\gamma\epsilon}\left[\left(3t^2+v^2\right)\right]\end{aligned} \tag{56}$$

Then the stress energy of the plate P

$$U^{\mathcal{S}}_K=\int_{P_0}\Phi(\mathcal{S})\,\mathrm{d}a, \tag{57}$$

where P_0 is the internal domain of the middle plane of the plate P.

4.2. The Cosserat Plate kinetic Energy Density

We define the plate stress energy density by the formula;

$$\Upsilon_C = \frac{h}{4}\int_{-1}^{1}\Upsilon_C\left\{\frac{\partial \boldsymbol{u}}{\partial t},\frac{\partial \boldsymbol{\varphi}}{\partial t}\right\}\mathrm{d}\zeta_3. \tag{58}$$

Taking into account the kinematics assumptions and integrating $\Upsilon\left\{\frac{\partial \boldsymbol{p}}{\partial t},\frac{\partial \boldsymbol{q}}{\partial t}\right\}$ with respect ζ_3 in $[-1,1]$ we obtain the explicit plate stress energy density expression in the form:

$$\Upsilon_C = \frac{h^3\rho}{24}\left(\frac{\partial \Psi_\alpha}{\partial t}\right)^2 + \frac{h\rho}{2}\left(\frac{\partial W}{\partial t}\right)^2 + \frac{4h\rho}{15}\left(\frac{\partial W^*}{\partial t}\right)^2 + \frac{2h\rho}{3}\frac{\partial W}{\partial t}\frac{\partial W^*}{\partial t}$$
$$+\frac{4hJ}{15}\left(\frac{\partial \Omega_\alpha^0}{\partial t}\right)^2 + \frac{hJ}{2}\left(\frac{\partial \hat{\Omega}_\alpha^0}{\partial t}\right)^2 + \frac{2hJ}{3}\frac{\partial \Omega_\alpha^0}{\partial t}\frac{\partial \hat{\Omega}_\alpha^0}{\partial t} + \frac{hJ}{6}\left(\frac{\partial \Omega_3}{\partial t}\right)^2.$$

Then the kinetic energy of the plate can be written

$$T_K^{\mathcal{S}} = \int_{P_0}\Upsilon_C\mathrm{d}a, \tag{59}$$

4.3. The Density of the Work Done over the Cosserat Plate Boundary

In the following consideration we also assume that the proposed stress, couple stress, and kinematic assumptions are valid for the lateral boundary of the plate P as well.

We evaluate the density of the work over the boundary $\Gamma_u \times [-h/2, h/2]$.

$$\mathcal{W}_1 = \frac{h}{2}\int_{-1}^{1}\left[\boldsymbol{\sigma_n}\cdot\boldsymbol{u} + \boldsymbol{\mu_n}\cdot\boldsymbol{\varphi}\right]\mathrm{d}\zeta_3. \tag{60}$$

Taking into account the stress and couple stress assumptions (26)-(34) and kinematic assumptions (42)-(45) we are able to represent $\mathcal{W}_1$ by the following expression:

$$\mathcal{W}_1 = \mathcal{S}_n\cdot\mathcal{U} = \breve{M}_\alpha\Psi_\alpha + \breve{Q}_\alpha^* W + \breve{Q}_\alpha^{94} W^* + \breve{R}_\alpha\Omega_\alpha^0 + \breve{R}_\alpha^*\hat{\Omega}_\alpha^0 + \breve{S}^*\Omega_3, \tag{61}$$

where the sets $\mathcal{S}_n$ and $\mathcal{U}$ are defined as

$$\mathcal{S}_n = \left[\breve{M}_\alpha, \breve{Q}^*, \breve{Q}_\alpha^{94}, \breve{R}_\alpha, \breve{R}_\alpha^*, \breve{S}^*\right],$$

$$\mathcal{U} = \left[\Psi_\alpha, W, W^*, \Omega_\alpha^0, \hat{\Omega}_\alpha, \Omega_3\right],$$

and

$$\breve{M}_\alpha = M_{\alpha\beta}n_\beta, \breve{Q}^* = Q_\beta^* n_\beta, \breve{R}_\alpha = R_{\alpha\beta}n_\beta,$$

$$\breve{S}^* = S_\beta^* n_\beta, \breve{Q}^{94} = \breve{Q}_\beta^{94} n_\beta, \breve{R}_\alpha^* = \breve{R}_{\alpha\beta}^* n_\beta.$$

In the above n_β is the outward unit normal vector to Γ_u.

The density of the work over the boundary $\Gamma_\sigma \times [-h/2, h/2]$

$$\mathcal{W}_2 = \frac{h}{2}\int_{-1}^{1}\left(\sigma_{o\alpha}u_\alpha + m_{o\alpha}\varphi_\alpha\right)n_\alpha\mathrm{d}\zeta_3$$

can be presented in the form

$$\mathcal{W}_2 = \mathcal{S}_o\cdot\mathcal{U} = \Pi_{o\alpha}\Psi_\alpha + \Pi_{o3}W + \Pi_{o3}^* W^* + M_{o\alpha}\Omega_\alpha^0 + M_{o\alpha}^*\hat{\Omega}_\alpha + M_{o3}^*\Omega_3$$

where

$$M_{\alpha\beta}n_\beta = \Pi_{o\alpha}, R_{\alpha\beta}n_\beta = M_{o\alpha},$$
$$Q_\alpha^* n_\alpha = \Pi_{o3}, S_\alpha^* n_\alpha = M_{o3}^*, \tag{62}$$
$$\hat{Q}_\alpha n_\alpha = \Pi_{o3}^*, R_{\alpha\beta}^* n_\beta = M_{o\alpha}^*.$$

Now n_β is the outward unit normal vector to Γ_σ, and

$$\Pi_{o\alpha} = \left(\frac{h}{2}\right)^2 \int_{-1}^{1} \zeta_3 \sigma_{o\alpha} \mathrm{d}\zeta_3, \; M_{o\alpha} = \frac{h}{2}\int_{-1}^{1} \mu_{o\alpha} \mathrm{d}\zeta_3,$$

$$\Pi_{o3} = \frac{h}{2}\int_{-1}^{1} \left(\sigma_{o3} - \sigma_0\right) \mathrm{d}\zeta_3, \; M_{o3}^* = \frac{h}{2}\int_{-1}^{1} \left(\mu_{o3} - t n_3\right) \mathrm{d}\zeta_3,$$

$$\Pi_{o3}^* = \frac{h}{2}\int_{-1}^{1} \left(\sigma_{o3} - \sigma_0\right) \mathrm{d}\zeta_3, \; M_{o\alpha}^* = \frac{h}{2}\int_{-1}^{1} \mu_{o\alpha} \mathrm{d}\zeta_3. \tag{63}$$

We are able to evaluate the work done at the top and bottom of the Cosserat plate by using boundary conditions (22) and (24)

$$\int_{T\cup B} \left(\sigma_{o3} u_3 + m_{o3} \varphi_{o3}\right) n_3 \mathrm{d}a - \int_{P_0} \left(pW + v\Omega_3^0\right) \mathrm{d}a.$$

4.4. The Cosserat Plate Internal Work Density

Here we define the density of the work done by the stress and couple stress over the Cosserat strain field:

$$\mathcal{W}_3 = \frac{h}{2}\int_{-1}^{1} \left(\boldsymbol{\sigma}\cdot\boldsymbol{\gamma} + \boldsymbol{\mu}\cdot\boldsymbol{\chi}\right) \mathrm{d}\zeta_3. \tag{64}$$

Substituting stress and couple stress assumptions and integrating the expression (64) we obtain the following expression:

$$\mathcal{W}_3 = \mathcal{S}\cdot\mathcal{E} = M_{\alpha\beta} e_{\alpha\beta} + Q_\alpha \omega_\alpha + Q_{3\alpha}^* \omega_\alpha^* + \hat{Q}_\alpha \hat{\omega}_\alpha + R_{\alpha\beta} \tau_{\alpha\beta} + R_{\alpha\beta}^* \tau_{\alpha\beta}^* + S_\alpha^* \tau_{3\alpha}, \tag{65}$$

where $\mathcal{E}$ is the Cosserat plate strain set of the the weighted averages of strain and torsion tensors

$$\mathcal{E} = \left[e_{\alpha\beta}, \omega_\beta, \omega_\alpha^*, \hat{\omega}_\alpha, \tau_{3\alpha}, \tau_{\alpha\beta}, \tau_{\alpha\beta}^*\right].$$

4.5. The Alternate Density Form of the Kinetic Energy

Here we define the density of the kinetic energy:

$$\mathcal{W}_4 = \frac{h}{4}\int_{-1}^{1} \left(\boldsymbol{p}\cdot\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{q}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial t}\right) \mathrm{d}\zeta_3.$$

which can be presented in the form

$$\mathcal{W}_4 = \boldsymbol{P}\cdot\frac{\partial \mathcal{U}}{\partial t}$$

where

$$\mathcal{U} = \left[\Psi_\alpha, W, \Omega_3, \Omega_\alpha^0, W^*, \hat{\Omega}_\alpha^0\right]$$

5. Cosserat Plate HPR Dynamic Principle

Let $\mathcal{A}$ denote the set of all admissible states that satisfy the Cosserat plate strain-displacement relation (54) and let Θ be a HPR functional on $\mathcal{A}$ defined by

$$\Theta(s,\eta) = U_K^S + T_C^S - \int_{P_0} \left(\mathcal{S}\cdot\mathcal{E} + \boldsymbol{P}\cdot\frac{\partial \mathcal{U}}{\partial t} - P\cdot\mathcal{W} + v\Omega_3^0\right) \mathrm{d}a + \int_{\Gamma_\sigma} \mathcal{S}_o \cdot \left(\mathcal{U} - \mathcal{U}_o\right) \mathrm{d}s + \int_{\Gamma_u} \mathcal{S}_n \cdot \mathcal{U} \mathrm{d}s, \tag{66}$$

for every $s = [\mathcal{U}, \mathcal{E}, \mathcal{S}] \in \mathcal{A}$. Here $P = \left(\hat{p}_1, \hat{p}_2\right)$ and $\mathcal{W} = \left(W, W^*\right)$.

Then

$$\delta\Theta(s,\eta) = 0$$

is equivalent to the plate bending system of equations (A) and constitutive formulas (B) mixed problems.

A. The bending equilibrium system of equations:

$$M_{\alpha\beta,\alpha} - Q_\beta = I_o \frac{\partial^2 \Psi_\beta}{\partial t^2}, \tag{67}$$

$$Q^*_{\alpha,\alpha} + \hat{p}_1 = \rho_o \frac{\partial^2 W^*}{\partial t^2}, \tag{68}$$

$$R_{\alpha\beta,\alpha} + \varepsilon_{3\beta\gamma}\left(Q^*_\gamma - Q_\gamma\right) = I_{o\alpha} \frac{\partial^2 \Omega^0_\alpha}{\partial t^2}, \tag{69}$$

$$\varepsilon_{3\beta\gamma} M_{\beta\gamma} + S^*_{\alpha,\alpha} = J_3 \frac{\partial^2 \Omega_3}{\partial t^2}, \tag{70}$$

$$\hat{Q}_{\alpha,\alpha} + \hat{p}_2 = \rho_o \frac{\partial^2 W}{\partial t^2}, \tag{71}$$

$$R^*_{\alpha\beta,\alpha} + \varepsilon_{3\beta\gamma}\hat{Q}_\gamma = I_{o\alpha} \frac{\partial^2 \hat{\Omega}^0_\alpha}{\partial t^2}, \tag{72}$$

where $I_o = \frac{h^3}{12}\rho, \rho_o = \frac{2h}{3}\rho$, $I_{o\alpha} = \frac{5h}{6}J$, $J_3 = \frac{5h^3}{48}J$, $\widehat{p_1} = \eta p$ and $\widehat{p_2} = \frac{2}{3}(1-\eta)p$, with the resultant traction boundary conditions:

$$M_{\alpha\beta} n_\beta = \Pi_{o\alpha}, R_{\alpha\beta} n_\beta = M_{o\alpha}, \tag{73}$$

$$Q^*_\alpha n_\alpha = \Pi_{o3}, S^*_\alpha n_\alpha = \Upsilon_{o3}, \tag{74}$$

at the part Γ_σ and the resultant displacement boundary conditions

$$\Psi_\alpha = \Psi_{o\alpha}, W = W_o, \Omega^0_\alpha = \Omega^0_{o\alpha}, \Omega_3 = \Omega_{o3}, \tag{75}$$

at the part Γ_u.

The constitutive formulas have the following reverse form[1]:

$$M_{\alpha\alpha} = \frac{\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)}\Psi_{\alpha,\alpha} + \frac{\lambda\mu h^3}{6(\lambda+2\mu)}\Psi_{\beta,\beta} + \frac{(3p_1+5p_2)\lambda h^2}{30(\lambda+2\mu)}, \tag{76}$$

$$M_{\beta\alpha} = \frac{(\mu-\alpha)h^3}{12}\Psi_{\alpha,\beta} + \frac{(\mu+\alpha)h^3}{12}\Psi_{\beta,\alpha} + (-1)^{\alpha'}\frac{\alpha h^3}{6}\Omega_3, \tag{77}$$

$$R_{\beta\alpha} = \frac{5(\gamma-\epsilon)h}{6}\Omega^0_{\beta,\alpha} + \frac{5(\gamma+\epsilon)h}{6}\Omega^0_{\alpha,\beta}, \tag{78}$$

$$R_{\alpha\alpha} = \frac{10h\gamma(\beta+\gamma)}{3(\beta+2\gamma)}\Omega^0_{\alpha,\alpha} + \frac{5h\beta\gamma}{3(\beta+2\gamma)}\Omega^0_{\beta,\beta}, \tag{79}$$

$$R_{\beta\alpha} = \frac{2(\gamma-\epsilon)h}{3}\hat{\Omega}_{\beta,\alpha} + \frac{2(\gamma+\epsilon)h}{3}\hat{\Omega}_{\alpha,\beta}, \tag{80}$$

$$R_{\alpha\alpha} = \frac{8\gamma(\gamma+\beta)h}{3(\beta+2\gamma)}\hat{\Omega}_{\alpha,\alpha} + \frac{4\gamma\beta h}{3(\beta+2\gamma)}\hat{\Omega}_{\beta,\beta}, \tag{81}$$

$$Q_\alpha = \frac{5(\mu+\alpha)h}{6}\Psi_\alpha + \frac{5(\mu-\alpha)h}{6}W_{,\alpha} + \frac{2(\mu-\alpha)h}{3}W^*_{,\alpha} + (-1)^\beta\frac{5h\alpha}{3}\Omega^0_\beta + (-1)^\beta\frac{5h\alpha}{3}\hat{\Omega}_\beta, \tag{82}$$

$$Q^*_\alpha = \frac{5(\mu-\alpha)h}{6}\Psi_\alpha + \frac{5(\mu-\alpha)^2 h}{6(\mu+\alpha)}W_{,\alpha} + \frac{2(\mu+\alpha)h}{3}W^*_{,\alpha} + (-1)^\alpha\frac{5h\alpha}{3}\left(\Omega^0_\beta + \frac{(\mu-\alpha)}{(\mu+\alpha)}\hat{\Omega}_\beta\right), \tag{83}$$

[1]In the following formulas a subindex $\beta = 1$ if $\alpha = 2$ and $\beta = 2$ if $\alpha = 1$.

$$\hat{Q}_\alpha = \frac{8\alpha\mu h}{3(\mu+\alpha)} W_{,\alpha} + (-1)^\alpha \frac{8\alpha\mu h}{3(\mu+\alpha)} \hat{\Omega}_\beta, \tag{84}$$

$$S_\alpha^* = \frac{5\gamma\epsilon h^3}{3(\gamma+\epsilon)} \Omega_{3,\alpha}, \tag{85}$$

$$N_{\alpha\alpha} = \frac{4\mu(\lambda+\mu)h}{\lambda+2\mu} U_{a,\alpha} + \frac{2\mu\lambda h}{\lambda+2\mu} U_{\beta,\beta} + \frac{\lambda h}{\lambda+2\mu}\sigma_0, \tag{86}$$

$$N_{\beta\alpha} = (\mu-\alpha) h U_{\beta,\alpha} + (\mu+\alpha) h U_{\alpha,\beta} + (-1)^\alpha 2\alpha h \Omega_3^0, \tag{87}$$

$$M_\alpha^* = \frac{4\gamma\epsilon h}{\gamma+\epsilon} \Omega_{3,\alpha}^0, \tag{88}$$

Proof of the principle. The variation of $\Theta(s)$

$$\delta\Theta(s) = \delta U_K^S(\eta) + \delta T_C^S(\eta) + \int_{P_0} \left\{ (\mathcal{K}[\mathcal{S}] - \mathcal{E}) \cdot \delta\mathcal{S} + \left(\mathcal{K}\left[\frac{\partial \mathcal{U}}{\partial t}\right] - \boldsymbol{P} \right) \cdot \delta \frac{\partial \mathcal{U}}{\partial t} - \mathcal{S}\delta\mathcal{E} \right.$$

$$\left. - \frac{\partial \mathcal{U}}{\partial t} \cdot \delta \boldsymbol{P} + P\delta W + v\delta\Omega_3^0 \right\} \mathrm{d}a + \int_{\Gamma_\sigma} \left\{ \delta\mathcal{S}_o \cdot (\mathcal{U} - \mathcal{U}_o) + \mathcal{S}_o \cdot \delta\mathcal{U} \right\} \mathrm{d}s + \int_{\Gamma_u} \mathcal{S}_n \cdot \delta\mathcal{U} \mathrm{d}s,$$

where

$$\mathcal{E} = \mathcal{K}[\mathcal{S}] = \mathcal{K} \cdot \mathcal{S},$$

where we call $\mathcal{K}$ the compliance Cosserat plate tensor.

We apply Green's theorem and integration by parts for $\mathcal{S}$ and $\delta\mathcal{U}$ [23] to the expression:

$$\int_{P_0} \left(\mathcal{S} \cdot \delta\mathcal{E} + \delta\boldsymbol{P} \cdot \frac{\partial \mathcal{U}}{\partial t} \right) \mathrm{d}a = \int_{\partial P_0} \mathcal{S}_o \delta \cdot \mathcal{U} \mathrm{d}s - \int_{P_0} \left\{ \left(M_{\alpha\beta,\alpha} - Q_\beta - I_o \frac{\partial^2 \Psi_\beta}{\partial t^2} \right) \delta\Psi_\beta + \left(Q_{\alpha,\alpha}^* - \rho_o \frac{\partial^2 W^*}{\partial t^2} \right) \delta W^* \right.$$

$$+ \left(R_{\alpha\beta,\alpha} + \varepsilon_{3\beta\gamma} (Q_\gamma^* - Q_\gamma) R_{\alpha\beta,\alpha} - I_{o\alpha} \frac{\partial^2 \Omega_\alpha^0}{\partial t^2} \right) \delta\Omega_\beta^0 + \left(Q_{\alpha,\alpha}^* - \rho_o \frac{\partial^2 W}{\partial t^2} \right) \delta W$$

$$+ \left(S_{\alpha,\alpha}^* + \varepsilon_{3\beta\gamma} M_{\beta\gamma} - J_3 \frac{\partial^2 \Omega_3}{\partial t^2} \right) \delta\Omega_3 + \left(N_{\alpha\beta,\alpha} - \rho_o \frac{\partial^2 W}{\partial t^2} \right) \delta U_\beta$$

$$\left. + \left(M_{\alpha,\alpha}^* + \varepsilon_{3\beta\gamma} N_{\beta\gamma} - I_{o\alpha} \frac{\partial^2 \hat{\Omega}_\alpha^0}{\partial t^2} \right) \delta\Omega_3^0 \right\} \mathrm{d}a.$$

Then based on the fact that $\delta\mathcal{U}$ and $\delta\mathcal{E}$ satisfy the Cosserat plate strain-displacement relation (54), we obtain

$$\delta\Theta(s,\eta) = \delta U_K^S(\eta) + \delta T_C^S(\eta) + \int_{P_0} \left\{ (\mathcal{K}[\mathcal{S}] - \mathcal{E}) \cdot \delta\mathcal{S} - \mathcal{S}\delta\mathcal{E} + \left(\mathcal{S}\left[\frac{\partial \mathcal{U}}{\partial t}\right] - \boldsymbol{P} \right) \cdot \delta \frac{\partial \mathcal{U}}{\partial t} - \frac{\partial \mathcal{U}}{\partial t} \cdot \delta \boldsymbol{P} \right\} \mathrm{d}a$$

$$+ \int_{P_0} \left\{ \left(M_{\alpha\beta,\alpha} - Q_\beta - I_o \frac{\partial^2 \Psi_\beta}{\partial t^2} \right) \delta\Psi_\beta + \left(Q_{\alpha,\alpha}^* + \hat{p}_1 - \rho_o \frac{\partial^2 W^*}{\partial t^2} \right) \delta W \right.$$

$$+ \left(\hat{Q}_{\alpha,\alpha} + \hat{p}_2 - \rho_o \frac{\partial^2 W}{\partial t^2} \right) \delta W + \left(R_{\alpha\beta,\alpha} + \varepsilon_{3\beta\gamma} (Q_\gamma^* - Q_\gamma) R_{\alpha\beta,\alpha} - I_{o\alpha} \frac{\partial^2 \Omega_\alpha^0}{\partial t^2} \right) \delta\Omega_\beta^0$$

$$+ \left(R_{\alpha\beta,\alpha}^* + \varepsilon_{3\beta\gamma} \hat{Q}_\gamma \right) \delta\Omega_\beta^0 + \left(S_{\alpha,\alpha}^* + \varepsilon_{3\beta\gamma} M_{\beta\gamma} - J_3 \frac{\partial^2 \Omega_3}{\partial t^2} \right) \delta\Omega_3$$

$$\left. + \left(N_{\alpha\beta,\alpha} - \rho_o \frac{\partial^2 W}{\partial t^2} \right) \delta U_\beta + \left(M_{\alpha,\alpha}^* + \varepsilon_{3\beta\gamma} N_{\beta\gamma} - I_{o\alpha} \frac{\partial^2 \hat{\Omega}_\alpha^0}{\partial t^2} \right) \delta\Omega_3^0 \right\} \mathrm{d}a$$

$$+ \int_{\Gamma_\sigma} \delta\mathcal{S}_o \cdot (\mathcal{U} - \mathcal{U}_o) \mathrm{d}s + \int_{\Gamma_u} (\mathcal{S}_o - \mathcal{S}_n) \cdot \delta\mathcal{U} \mathrm{d}s.$$

If s is a solution of the mixed problem, then

$$\delta\Theta(s,\eta)=0$$

On the other hand, some extensions of the fundamental lemma of calculus of variations [23] together with the fact that $\mathcal{U}$ and $\mathcal{E}$ satisfy the Cosserat plate strain-displacement relation (54) imply that $\mathcal{S}$ is a solution of the A and B mixed problems. The uniqueness proof is similar to [20].

Remark. The above equilibrium equations and boundary conditions for the Cosserat plate can also be obtained by substituting polynomial approximations of stress and couple stress directly to the elastic equilibrium (1)-(2) and the boundary conditions (22)-(25) and collecting and equating to zero all coefficients of the resulting polynomials with respect to variable x_3.

6. Micropolar Plate Dynamic Field Equations

In order to obtain the micropolar plate bending field equations in terms of the kinematic variables, we substitute the constitutive formulas in the reverse form (76)-(88) into the bending system of Equations (67)-(72). The micropolar plate bending field equations can be written in the following form:

$$L\mathcal{U}=K\frac{\partial^2\mathcal{U}}{\partial t^2}+\mathcal{F}(\eta) \tag{89}$$

where

$$L=\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & 0 & L_{16} & kL_{13} & 0 & L_{16} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{16} & 0 & kL_{23} & L_{16} & 0 \\ -L_{13} & -L_{23} & L_{33} & 0 & L_{35} & L_{36} & L_{77} & L_{38} & L_{39} \\ L_{41} & L_{42} & 0 & L_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & -L_{16} & -L_{38} & 0 & L_{55} & L_{56} & -kL_{35} & L_{58} & 0 \\ L_{16} & 0 & -L_{39} & 0 & L_{56} & L_{66} & -kL_{36} & 0 & L_{58} \\ -L_{13} & -L_{14} & L_{73} & 0 & L_{35} & L_{36} & L_{77} & L_{78} & L_{79} \\ 0 & -L_{16} & -L_{78} & 0 & L_{85} & L_{56} & -kL_{35} & L_{88} & kL_{56} \\ L_{16} & 0 & -L_{79} & 0 & L_{56} & L_{55} & -kL_{36} & kL_{56} & L_{99} \end{bmatrix},$$

$$K=\begin{bmatrix} \frac{h^3}{12}\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h^3}{12}\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h\rho & 0 & 0 & 0 & \frac{2h}{3}\rho & 0 & 0 \\ 0 & 0 & 0 & hJ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8h}{15}J & 0 & 0 & \frac{2h}{3}J & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{8h}{15}J & 0 & 0 & \frac{2h}{3}J \\ 0 & 0 & \frac{2h}{3}\rho & 0 & 0 & 0 & \frac{8h}{15}\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2h}{3}J & 0 & 0 & hJ & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2h}{3}J & 0 & 0 & hJ \end{bmatrix},$$

$$\mathcal{U} = \begin{bmatrix} \Psi_1 & \Psi_2 & W & \Omega_3 & \Omega_1^0 & \Omega_2^0 & W^* & \Omega_1^0 & \Omega_2^0 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{F}(\eta) = \begin{bmatrix} -\frac{3h^2\lambda(3p_{1,1}+5p_{2,1})}{30(\lambda+2\mu)} & -\frac{3h^2\lambda(3p_{1,2}+5p_{2,2})}{30(\lambda+2\mu)} & -p_1 & 0 & 0 & 0 & \frac{h^2(3p_1+4p_2)}{24} & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$p_1 = \eta p,$$

$$p_2 = \frac{(1-\eta)}{2} p$$

The operators L_{ij} are defined as follows

$$L_{11} = c_1 \frac{\partial^2}{\partial x_1^2} + c_2 \frac{\partial^2}{\partial x_2^2} - c_3, L_{12} = (c_1 - c_2)\frac{\partial^2}{\partial x_1 x_2}, L_{13} = c_{11}\frac{\partial}{\partial x_1}, L_{14} = c_{12}\frac{\partial}{\partial x_2},$$

$$L_{16} = c_{13}, L_{17} = kc_{11}\frac{\partial}{\partial x_1}, L_{22} = c_2\frac{\partial^2}{\partial x_1^2} + c_1\frac{\partial^2}{\partial x_2^2} - c_3, L_{23} = c_{11}\frac{\partial}{\partial x_2}, L_{24} = -c_{12}\frac{\partial}{\partial x_1},$$

$$L_{33} = c_4\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right), L_{35} = -c_{13}\frac{\partial}{\partial x_2}, L_{36} = c_{13}\frac{\partial}{\partial x_1}, L_{38} = -c_{10}\frac{\partial}{\partial x_2}, L_{39} = c_{10}\frac{\partial}{\partial x_1},$$

$$L_{41} = -c_{12}\frac{\partial}{\partial x_2}, L_{42} = c_{12}\frac{\partial}{\partial x_1}, L_{44} = c_6\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2c_{12},$$

$$L_{55} = c_7\frac{\partial^2}{\partial x_1^2} + c_8\frac{\partial^2}{\partial x_2^2} - 2c_{13}, L_{56} = (c_7 - c_8)\frac{\partial^2}{\partial x_1 x_2}, L_{58} = -c_9,$$

$$L_{66} = c_8\frac{\partial^2}{\partial x_1^2} + c_7\frac{\partial^2}{\partial x_2^2} - 2c_{13}, L_{73} = c_5\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right), L_{77} = c_3\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right),$$

$$L_{78} = -c_{14}\frac{\partial}{\partial x_2}, L_{79} = c_{14}\frac{\partial}{\partial x_1}, L_{85} = c_7\frac{\partial^2}{\partial x_1^2} + c_8\frac{\partial^2}{\partial x_2^2} - 2c_{13},$$

$$L_{88} = kc_7\frac{\partial^2}{\partial x_1^2} + kc_8\frac{\partial^2}{\partial x_2^2} - c_{15}, L_{99} = kc_8\frac{\partial^2}{\partial x_1^2} + kc_7\frac{\partial^2}{\partial x_2^2} - c_{15}$$

where

$$k = \frac{4}{5}, c_1 = \frac{h^3\mu(\lambda+\mu)}{3(\lambda+2\mu)}, c_2 = \frac{h^3(\alpha+\mu)}{12}, c_3 = \frac{5h(\alpha+\mu)}{6}, c_4 = \frac{5h(\alpha-\mu)^2}{6(\alpha+\mu)},$$

$$c_5 = \frac{h(5\alpha^2+6\alpha\mu+5\mu^2)}{6(\alpha+\mu)}, c_6 = \frac{h^3\gamma\epsilon}{3(\gamma+\epsilon)}, c_7 = \frac{10h\gamma(\beta+\gamma)}{3(\beta+2\gamma)}, c_8 = \frac{5h(\gamma+\epsilon)}{6},$$

$$c_9 = \frac{10h\alpha^2}{3(\alpha+\mu)}, c_{10} = \frac{5h\alpha(\alpha-\mu)}{3(\alpha+\mu)}, c_{11} = \frac{5h(\alpha-\mu)}{6}, c_{12} = \frac{h^3\alpha}{6},$$

$$c_{13} = \frac{5h\alpha}{3}, c_{14} = \frac{h\alpha(5\alpha+3\mu)}{3(\alpha+\mu)}, c_{15} = \frac{2h\alpha(5\alpha+4\mu)}{3(\alpha+\mu)}.$$

The right-hand side, and therefore the solution $\mathcal{U}$ are the functions of the splitting parameter η. The optimal value of the parameter η corresponds to the minimum of elastic energy over the micropolar strain field. The algorithm was described in [21]. The system (89) should be complemented with the boundary conditions. We describe the case of the hard simply supported boundary conditions in the numerical simulation section.

Numerical Simulation

Let us consider a square plate $a \times a$ of thickness h. In our numerical computations we will consider a plate of thickness h = 0.1 m made of polyurethane foam, and the ratio a/h varying from 5 to 30. The values of the technical elastic parameters for the polyurethane foam are reported in [1]: $E = 299.5$ MPa, $\nu = 0.44$, $l_t = 0.62$ mm, $l_b = 0.327$, $N^2 = 0.04$. These values correspond to the following values of Lamé and asymmetric parameters: $\lambda = 762.616$, $\mu = 103.993$, $\alpha = 4.333$, $\beta = 39.975$, $\gamma = 39.975$, $\epsilon = 4.505$ (the ratio β/γ is equal to 1 for the bending).

The boundary $G = G_1 \cup G_2$ is given by

$$G_1 = \left\{ (x_1, x_2) : x_1 \in \{0, a\}, x_2 \in [0, a] \right\}$$

$$G_2 = \left\{ (x_1, x_2) : x_2 \in \{0, a\}, x_1 \in [0, a] \right\}$$

and the hard simply supported boundary conditions can be represented in the following mixed Dirichlet-Neumann form [21]:

$$G_1 : W = 0, W^* = 0, \Psi_2 = 0, \Omega_1^0 = 0, \hat{\Omega}_1^0 = 0;$$

$$G_1 : \Omega_3 = 0, \frac{\partial \Psi_1}{\partial n} = 0, \frac{\partial \Omega_2^0}{\partial n} = 0, \frac{\partial \hat{\Omega}_2^0}{\partial n} = 0;$$

$$G_2 : W = 0, W^* = 0, \Psi_1 = 0, \Omega_2^0 = 0, \hat{\Omega}_2^0 = 0;$$

$$G_2 : \Omega_3 = 0, \frac{\partial \Psi_2}{\partial n} = 0, \frac{\partial \Omega_1^0}{\partial n} = 0, \frac{\partial \hat{\Omega}_1^0}{\partial n} = 0.$$

By applying the method of separation of variables for the two-dimensional eigenvalue problem (89) with the hard simply supported boundary conditions we obtain the kinematic variables in the following form:

$$\begin{gathered}
\Psi_1^{nm} = A_1 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \Psi_2^{nm} = A_2 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \\
W^{nm} = A_3 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \ \Omega_3^{nm} = A_4 \cos\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \\
\Omega_1^{0,nm} = A_5 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \Omega_2^{0,nm} = A_6 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \\
W^{*,nm} = A_7 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \hat{\Omega}_1^{0,nm} = A_8 \sin\left(\frac{n\pi x_1}{a}\right) \cos\left(\frac{m\pi x_2}{a}\right) \sin(\omega t), \\
\hat{\Omega}_2^{0,nm} = A_9 \cos\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{a}\right) \sin(\omega t).
\end{gathered} \tag{90}$$

and a standard eigenvalue problem for a system of 9 algebraic equations. Thus the model produces a spectrum of 9 infinite sequences $\{\omega_i^{nm}\}$ $(i = 1, 2, \cdots, 9; n, m \in \mathbb{N})$ of eigenfrequencies, which are related to the rotatory and flexural vibrations, particularly ω_1^{nm} and ω_2^{nm} correspond to the rotatory vibration of the middle plane, ω_3^{nm} —flexural vibration, ω_7^{nm} —transverse variation of flexural vibration, ω_4^{nm}, ω_5^{nm}, ω_6^{nm} —microrotatory vibration, and ω_8^{nm}, ω_9^{nm} —transverse variation of microrotatory vibration.

Preliminary computations show that only $\omega_1^{nm}, \omega_3^{nm}$ eigenfrequencies remain nonzero when asymmetric constants of Cosserat elasticity tend to zero. These frequencies corresponds to the free oscillation of the case of Mindlin-Reissner Plate. We can see that for the case the first iegenfrequencies $\{\omega_i^{11}\}$ in **Figures 1-3**.

We perform computations for different levels of the asymmetric microstructure by reducing the values of the elastic asymmetric parameters. **Figure 4** illustrates the typical size effect of micropolar dynamic plate theory that predicts that micropolar plates made of smaller thickness has higher eigenfrequencies that would be expected on the basis of the Midlin-Reissner plate theory. Similar experimental behavior was reported in [1] for

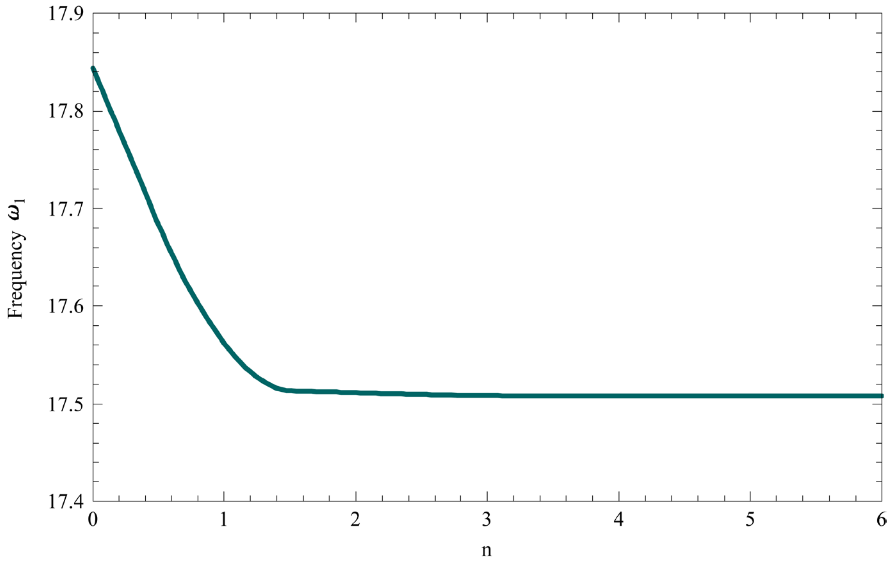

Figure 1. Dependence of $\omega_1 = \omega_1^{11}$ on the 10^{-n} of the original values of asymmetric constants.

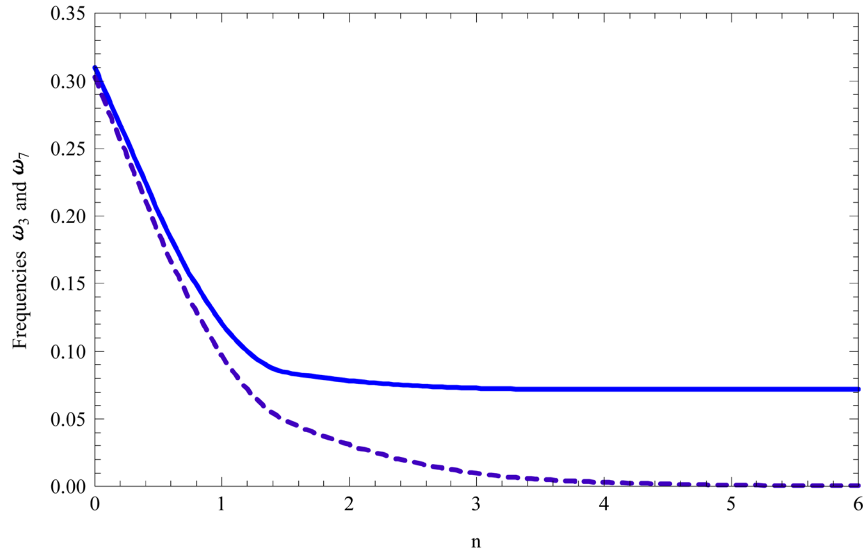

Figure 2. Dependence of $\omega_3 = \omega_3^1$ (solid line) and $\omega_7 = \omega_7^1$ (dashed line) on the 10^{-n} of the original values of asymmetric constants.

torsion and bending of cylindrical rods of a Cosserat solid.

We also check how the total energy of the free oscillation depends of the value of the shear correction factor κ in the constitutive formulas:

$$Q_\alpha = \kappa\frac{(\mu+\alpha)h}{6}\Psi_\alpha + \kappa\frac{(\mu-\alpha)h}{6}W_{,\alpha} + \frac{2(\mu-\alpha)h}{3}W^*_{,\alpha} + (-1)^\beta\frac{\kappa h\alpha}{3}\Omega^0_\beta + (-1)^\beta\frac{\kappa h\alpha}{3}\hat{\Omega}_\beta, \tag{91}$$

$$Q^*_\alpha = \kappa\frac{(\mu-\alpha)h}{6}\Psi_\alpha + \kappa\frac{(\mu-\alpha)^2 h}{6(\mu+\alpha)}W_{,\alpha} + \frac{2(\mu+\alpha)h}{3}W^*_{,\alpha} + (-1)^\alpha\frac{\kappa h\alpha}{3}\left(\Omega^0_\beta + \frac{(\mu-\alpha)}{(\mu+\alpha)}\hat{\Omega}_\beta\right). \tag{92}$$

We consider a rectangular plate $a \times b$ of thickness $h = 0.1\text{ m}$. As we can see from **Figure 5**, the minimum value of the shear correction factor κ is shifted to the left by 4% - 5% depending on the geometry of the

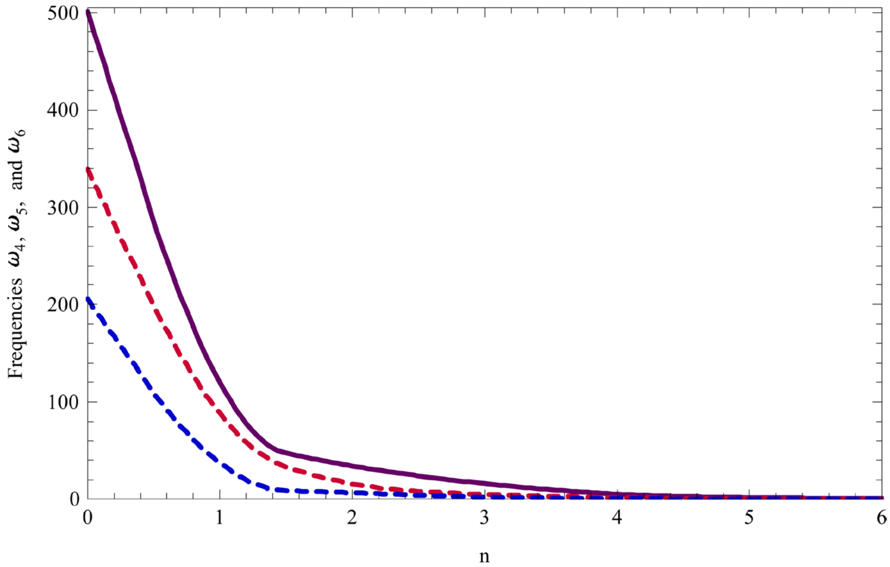

Figure 3. Dependence of $\omega_4 = \omega_4^{11}$ (solid line), $\omega_5 = \omega_5^{11}$ (dashed red line) and $\omega_6 = \omega_6^{11}$ (dashed blue line) on the 10^{-n} of the original values of asymmetric constants.

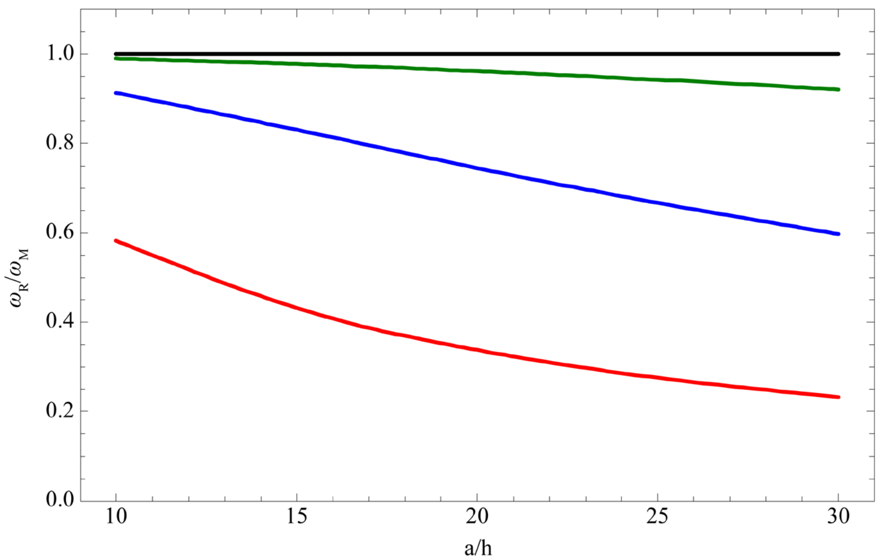

Figure 4. Comparison of the frequencies ω_R of the Mindlin-Reissner and ω_M of micropolar plates, which illustrates the influence of the thickness of the micropolar plate (red line—micropolar plate; blue line—micropolar plate with 10%; green line—micropolar plate with 1% of the original values of asymmetric constants; black line—Mindlin-Reissner plate).

plates. This result is consistent with the changes of this factor used in the Mindlin-Reissner plate theory.

7. Conclusion

This paper presents a mathematical model for the vibration of micropolar elastic plates. This model is based on the proposed generalization of Hellinger-Prange-Reissner (HPR) variational principle for the linearized micropolar (Cosserat) elastodynamics. The modeling of the plate vibration is based on the HPR variational principle for the dynamics of Cosserat plates, which incorporates most of assumptions of the authors' enhanced mathematical model for Cosserat plate deformation. The dynamic theory of the plates obtained from the dynamic

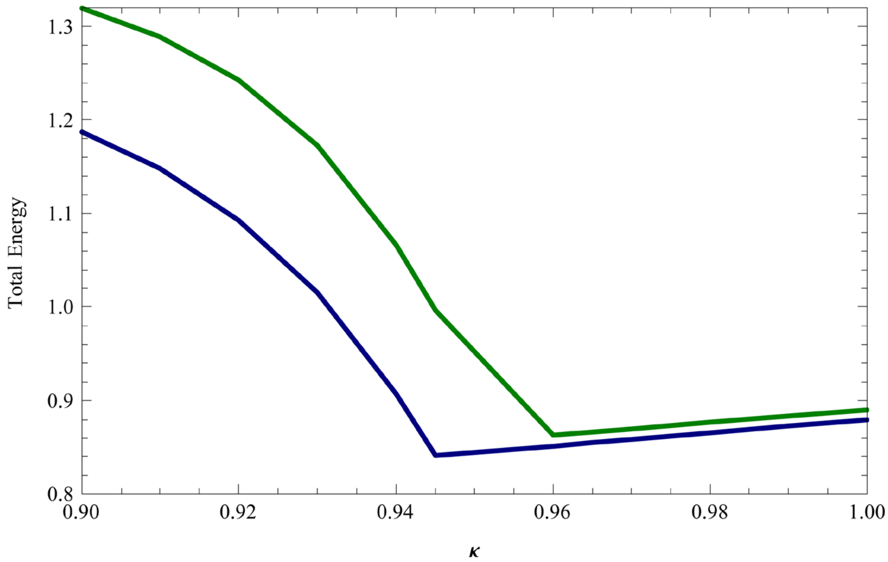

Figure 5. Total energy distribution with respect to the shear correction factor κ (blue line—micropolar plate 3.0 m × 3.0 m; green line—micropolar plate 3.0 m × 4.0 m).

variational principle includes a system of dynamics equations and the constitutive relations. The preliminary computations of the rectangular plate vibration predict additional natural frequencies, which are related with the material microstructure and obey the size-effect principle similar to the known from the micropolar plate deformation. The computations also show how natural frequencies of micropolar plate converge to classic Mindlin-Reissner plates and the total vibration energy can get 4% - 5% smaller depending on a parameter in the constitutive formulas and the geometry of the plates. These result are consistent with the modification (from $5/6$ to $\pi^2/12$) of the similar parameter used in the Mindlin-Reissner plate theory.

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