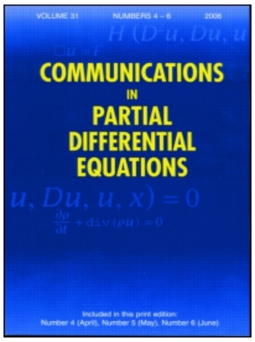

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Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces

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ABSTRACT
We consider a transmission problem consisting of two semilinear parabolic equations involving fractional diffusion operators of different orders in a general non-smooth setting with emphasis on Lipschitz interfaces and transmission conditions along the interface. We give a unified framework for the existence and uniqueness of strong and mild solutions, and their global regularity properties.

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1. Introduction

Anomalous diffusion is of great interest to science. A number of stochastic models for explaining anomalous diffusion have been introduced in the literature; among them we quote the fractional Brownian motion, the continuous time random walk, the Lévy flight, the Schneider–Grey Brownian motion and, more generally, random walk models based on evolution equations of single and distributed fractional order in time and/or in space [11, 20, 38, 47, 49]. To be more precise, let $\mathcal{K}: \mathbb{R}^N \to [0,\infty)$ be an even function such that

$$\sum_{k\in\mathbb{Z}^N} \mathcal{K}(k) = 1. \tag{1.1}$$

Given a small $h > 0$, we consider a random walk on the lattice $h\mathbb{Z}^N$. We suppose that at any unit time τ (which may depend on h) a particle jumps from any point of $h\mathbb{Z}^N$ to any other point. The probability for which a particle jumps from a point $hk \in h\mathbb{Z}^N$ to the point $h\tilde{k}$ is taken to be $\mathcal{K}(k-\tilde{k}) = \mathcal{K}(\tilde{k}-k)$. Note that, differently from the standard random walk, in this process the particle may experience arbitrarily long jumps, though with small probability. Let $u(x,t)$ be the probability that our particle lies at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$. Then $u(x,t+\tau)$ is the sum of all the probabilities of the possible positions $x+hk$ at time t weighted by the probability of jumping from $x+hk$ to x. That is,

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^N} \mathcal{K}(k)u(x+hk,t).$$

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Using (1.1) we have the evolution law:

$$u(x,t+\tau)-u(x,t)=\sum_{k\in\mathbb{Z}^N}\mathcal{K}(k)\left[u(x+hk,t)-u(x,t)\right]. \tag{1.2}$$

In particular, in the case when $\tau=h^{2s}$ and $\mathcal{K}$ is homogeneous (i.e., $\mathcal{K}(y)=|y|^{-(N+2s)}$ for $y\neq 0$, $\mathcal{K}(0)=0$, and $0<s<1$), (1.1) holds and $\mathcal{K}(k)/\tau=h^N\mathcal{K}(hk)$. Therefore, we can rewrite (1.2) as follows:

$$\frac{u(x,t+\tau)-u(x,t)}{\tau}=h^N\sum_{k\in\mathbb{Z}^N}\mathcal{K}(hk)\left[u(x+hk,t)-u(x,t)\right]. \tag{1.3}$$

Notice that the term on the right-hand side of (1.3) is just the approximating Riemann sum of

$$\int_{\mathbb{R}^N}\mathcal{K}(y)\left[u(x+y,t)-u(x,t)\right]dy.$$

Thus letting $\tau=h^{2s}\to 0^+$ in (1.3), we obtain

$$\partial_t u(x,t)=\int_{\mathbb{R}^N}\frac{u(x+y,t)-u(x,t)}{|y|^{N+2s}}dy. \tag{1.4}$$

The integral on the right-hand side of (1.4) has a singularity at $y=0$. However when $0<s<1$ and u is smooth and bounded, such integral is well defined as a principal value, that is,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^N\setminus B(0,\varepsilon)}\frac{u(x+y,t)-u(x,t)}{|y|^{N+2s}}dy&=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^N\setminus B(x,\varepsilon)}\frac{u(z,t)-u(x,t)}{|z-x|^{N+2s}}dz\\ &=-\left(C_{N,s}\right)^{-1}(-\Delta)^s u(x,t),\end{aligned} \tag{1.5}$$

for a proper normalizing constant $C_{N,s}>0$ (see (1.6) and Section 2). This shows that a simple random walk with possibly long jumps produces at the limit a singular integral with a homogeneous kernel. For more details on this topic we refer to [49]. In the case when in (1.5), $\mathbb{R}^N$ is replaced by an arbitrary open set $G\subset\mathbb{R}^N$ and the integral kernel is restricted only to the open set, we formally obtain the so-called *regional fractional Laplacian* $-(-\Delta)^s_G$ (cf. [3, 5, 22–24]). More precisely, for

$$u\in\mathcal{L}^1_s(G)=\left\{u:\ G\to\mathbb{R}\ \text{ measurable},\ \int_G\frac{|u(x)|}{(1+|x|)^{N+2s}}dx<\infty\right\},$$

$x\in G$ and $\varepsilon>0$, we let

$$(-\Delta)^s_{G,\varepsilon}u(x)=C_{N,s}\int_{\{y\in G,|y-x|>\varepsilon\}}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,\qquad C_{N,s}=\frac{s2^{2s}\Gamma_0\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma_0(1-s)}, \tag{1.6}$$

where Γ_0 denotes the usual Gamma function. Define

$$(-\Delta)^s_G u(x)=C_{N,s}\text{P.V.}\int_G\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy=\lim_{\varepsilon\downarrow 0}(-\Delta)^s_{G,\varepsilon}u(x),\quad x\in G,$$

provided that the limit exists. With the latter definition, the evolution law

$$\partial_t u+(-\Delta)^s_G u=0 \tag{1.7}$$

corresponds to a kind of "censored" stable process in $G \subset \mathbb{R}^N$, which is a Lévy motion "forced" to stay inside G. Such "restricted" Lévy motions show up an important models in both applied mathematics and applied probability [3, 5, 11, 20, 29, 40, 46], as well as in models in biology and ecology [27, 45]. It is interesting to note that the fractional heat equations (1.4), (1.7) also emerge as the hydrodynamic limit of interacting particle systems that are superdiffusive in nature, that is, the limit of systems on which particles may perform long jumps in the context of Lévy processes [29]. A great deal of mathematical literature has grown in the last decade for such parabolic problems with anomalous diffusion. Such a description has been undertaken in some detail in [15, 17] (and the references contained therein), and we shall not attempt to describe it here. In this paper, we exclusively focus on work that is directly related to aspects of transmission phenomena for problems of form similar to (1.7).

It is well known that the study of transmission problems refers to the analysis of models involving different media and/or different physical laws in separate regions of a given (fixed) domain Ω. Such transmission problems involve classical "diffusion" operators, typically, the usual Laplacian Δ, and are relevant for a wide range of problems in material science, physics and biology. Classical examples include transmission problems associated with electromagnetic, thermodynamic (or heat conduction) processes in (disjoint) regions with different conductivity constants, which are separated by a thin (possibly non-smooth) interface. We refer the reader to the book [4] for a fairly large description of the current literature on this class of problems on smooth domains. More recently, the case of non-smooth domains and/or various classes of transmission conditions (including dynamic transmission conditions possessing sources along the interface) has also been addressed in [1, 2, 10, 12, 14, 18, 19, 26, 28, 33–36, 42, 48]. To the best of our knowledge, the study of transmission problems involving diffusion operators of fractional type has not yet been considered in the literature with the exception of [31, 32]. In these references, a local-nonlocal elliptic transmission-like problem in $\mathbb{R}^N$, $N \geq 2$, is formulated such that $\mathbb{R}^N$ is divided into two disjoint open sets Ω_1, Ω_2, which are separated by a (sufficiently) smooth interface Σ; on the side Ω_1 the function satisfies a nonlocal linear elliptic equation (involving the usual fractional Laplacian $(-\Delta)^{1/2}$, see (2.1) below) and on the other Ω_2, it satisfies a local linear second-order elliptic problem (involving the usual diffusion operator $-\Delta$), while a nonlocal-local transmission condition is imposed along the interface Σ. The latter is interpreted by means of a given variational formulation which involves the use of fractional conormal derivatives also defined and investigated elsewhere in [9]. Such transmission problems are of intrinsic mathematical interest for their applications and interpretations in various contexts (for instance, in quasi-geostrophic dynamics). Beside the existence of (suitably weak) solutions for the corresponding linear transmission problem, the main goal of [31] is also to deduce their Hölder continuity following the method of De Giorgi. We refer the reader to [31, Theorem 1.1] and also [32, Section 4.3] for a continuity result that corresponds to the associated parabolic problem. Comparable continuity results for the corresponding elliptic problem when Σ is merely a Lipschitz graph can be also found in [31, Theorem 1.2]. When Σ is a flat surface, the qualitative behavior of the continuous solution near Σ is also investigated in [31, Theorems 1.3–1.4]. Further extensions to some nonlocal-nonlocal elliptic transmission problems are also discussed in [32, Chapter 4].

In order to formulate our problem properly, consider a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ and let Σ be such that $\Omega = \Omega_+ \cup \Sigma \cup \Omega_-$ with $\Sigma = \overline{\Omega}_+ \cap \overline{\Omega}_-$. That is, Ω is separated into two components Ω_+, Ω_- by a Lipschitz hypersurface Σ (see Figure 1). Since Σ is a

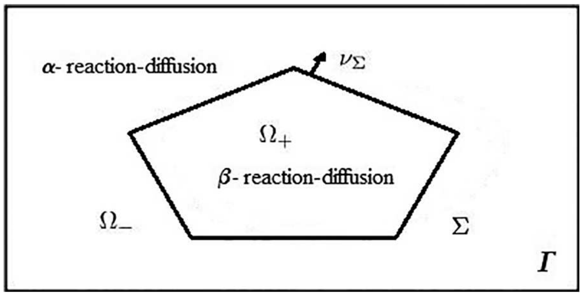

Figure 1. Transmission model with anomalous diffusion.

$(N-1)$-dimensional Lipschitz surface, we have that $\mathcal{H}^{N-1}(\Sigma) < \infty$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff (i.e., the surface) measure and we denote by σ_Σ (or simply by σ if there is no confusion) the restriction of $\mathcal{H}^{N-1}$ to the set Σ. We have that

$$\partial\Omega_+ = \Sigma \quad \text{and} \quad \partial\Omega_- = \Gamma \cup \Sigma.$$

The goal of this work is for the completion of a unified framework for a general class of nonlinear *nonlocal transmission problems*, of the form

$$\partial_t u_+ + (-\Delta)^{\beta}_{\Omega_+} u_+ + f_+(u_+) = 0, \quad \text{in } J \times \Omega_+, \tag{1.8}$$

$$\partial_t u_- + (-\Delta)^{\alpha}_{\Omega_-} u_- + f_-(u_-) = 0, \quad \text{in } J \times \Omega_-. \tag{1.9}$$

where $J = (0, T), f_+, f_-$ are nonlinear functions and $\alpha, \beta \in (1/2, 1)$ with $\alpha \geq \beta$. Equations (1.8) and (1.9) are coupled together with the transmission conditions

$$u_+ = wu_-, \quad \mathcal{N}_w(u_+, u_-) + \mathfrak{b}u_- = 0, \quad \text{on } J \times \Sigma. \tag{1.10}$$

Here, $\mathfrak{b} = \mathfrak{b}(x) \geq 0, \mathcal{N}_w(u_+, u_-)$ denotes the jump

$$\mathcal{N}_w(u_+(x), u_-(x)) := C_\beta w(x) \mathcal{N}^{2-2\beta} u_+(x) - C_\alpha \mathcal{N}^{2-2\alpha} u_-(x), \ x \in \Sigma,$$

where $w = w(x)$ is fixed, sufficiently smooth (say, of class $C^1(\Sigma)$), $\mathcal{N}^{2-2\beta}u_+$ and $\mathcal{N}^{2-2\alpha}u_-$, are the fractional normal derivative (see Section 2.2 below) of the function u_+ and u_-, respectively, and finally C_α, C_β are explicit constants depending only on α and β, respectively (see (2.14)). We also assume that $w \not\equiv 0$ σ a.e. on Σ.

We assume, for the sake of simplicity, that Γ is a Lipschitz hypersurface too. On Γ, we consider fractional Neumann boundary conditions, of the form

$$\mathcal{N}^{2-2\alpha}u_- = 0, \text{ on } J \times \Gamma, \tag{1.11}$$

although other (such as, Dirichlet or fractional Robin) boundary conditions on Γ may be used as well (see, for instance, [15]). Initial conditions for (1.8)–(1.11) must also be prescribed on $\Omega \backslash \Sigma$, that is,

$$u_+(\cdot, 0) = u_+^0 \text{ in } \Omega_+, \ u_-(\cdot, 0) = u_-^0 \text{ in } \Omega_-. \tag{1.12}$$

We note that our problem is essentially different from that of [31] (see Remark 2.1, as well as [32]) since it involves "restricted" fractional operators (i.e., the regional fractional Laplacian $(-\Delta)^{\alpha}_{G}$, (with $G = \Omega_+$ and/or Ω_-) of order $\alpha \in (1/2, 1)$) and the corresponding transmission conditions (1.10) involve a different kind of fractional normal derivative (see Section 2). Besides, our general setting (see Figure 1) requires only that Σ is merely Lipschitz continuous and the parabolic problem we investigate is also nonlinear.

We can summarize the main features of the present work as follows.

(a) To the best of our knowledge this is the first work to address the well-posedness of transmission problems associated with different orders of diffusion in Ω_+ and Ω_-, respectively. In particular, we view the framework developed here as one that extends our recent work from [15] to the case of different order of diffusion in Ω, when $\alpha \geq \beta$.

(b) We allow the nonlinear reactive forces $f_\pm = f_\pm(u_\pm)$ to have arbitrary growth without imposing any one-sided conditions, such as coercivity. We provide a complete analysis based on potential theory for the linear problem and on tools from semigroup theory to show the existence of properly defined mild solutions. The main results are presented in Theorem 3.3 (for local existence) and Theorem 3.7 (for local continuation and boundedness). Furthermore, we provide theorems concerning the existence of strong (differentiable) solutions that are bounded in $L^\infty(\Omega\backslash\Sigma)$ (see Theorem 4.2), and provide sufficient conditions for their global regularity (see Corollary 4.3). Finally, we also provide sufficient conditions and derive *explicit* uniform L^∞-estimates from some given L^r-estimate of the mild solution (see Theorems 4.4 and 4.5).

(c) The approach for the case $\alpha \geq \beta$ is inspired by a strategy developed in the book [44] for parabolic problems associated with second-order differential operators. However, many challenges had to be overcome to accommodate these cases. In particular, a complete study of the operator associated with (1.8)–(1.12) is critical in understanding both issues of well-posedness and the global regularity of solutions in L^p-like spaces ($p \in [1, \infty]$). More precisely, we show that a certain realization of $((-\Delta)^\beta_{\Omega_+}, (-\Delta)^\alpha_{\Omega_-})$ with suitable transmission conditions generates a submarkovian semigroup $S_{\alpha,\beta} = \{S_{\alpha,\beta}(t)\}_{t\geq 0}$, satisfying some crucial $L^q - L^r$-ultracontractivity estimates. Taking advantage of these properties we introduce a natural notion of integral solution for the corresponding nonlinear transmission problem associated with the initial datum $u_0 \in L^p(\Omega\backslash\Sigma)$ where $u_0 = u^0_+$ in Ω_+, $u_0 = u^0_-$ in Ω_-; we then investigate their local and global behavior and their further differentiability properties. This approach allows us to handle less regular solutions by extending the Hilbert-space approach used in [15, 17] to a Banach space framework where the initial datum can belong to any $L^p(\Omega\backslash\Sigma)$, for $p \in [1, \infty]$.

(d) We also address the case when the diffusion in Ω_- dominates the diffusion in Ω_+, i.e., when $\alpha \leq \beta$. We conclude with the validity of comparable results for the transmission problem (1.8) and (1.9), albeit with different transmission conditions (see Section 5). Here, the arguments follow the same proofs performed for the case $\alpha \geq \beta$.

(e) Our approach provides for a clear road map to treat more general transmission problems which are mixed in their order of diffusion, including local-nonlocal transmission problems. We refer the reader to Section 5 for further details.

Our main assumptions and results for the corresponding linear parabolic and elliptic problems are presented in Section 2. The main results are Theorems 2.16 and 2.22. The proofs of the well-posedness of the nonlinear system (1.8)–(1.12) are given in Section 3. The existence of strong (differentiable) solutions and their (global) regularity properties are established in Section 4. We conclude with Section 5 which is concerned with the remaining case of $\alpha \leq \beta$ and discuss the application of these techniques to other related transmission problems. We also give here a list of open problems.

2. The linear parabolic and elliptic problems

In this section we introduce the functional framework associated with the transmission problem in question and then derive semigroup type results for the linear operator corresponding to the linear problem. Existence and regularity of solutions to elliptic problems associated with the linear operator are also investigated. All these tools are necessary in the study of the nonlinear transmission problem (1.8)–(1.12).

2.1. The functional framework

The *fractional Laplacian* $(-\Delta)^s u$ of $u \in \mathcal{L}^1_s(\mathbb{R}^N)$ is defined by the singular integral

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy = \lim_{\varepsilon\downarrow 0}(-\Delta)^s_{\mathbb{R}^N,\varepsilon}u(x),\ x\in\mathbb{R}^N, \tag{2.1}$$

provided that the limit exists. We notice that if $0 < s < 1/2$ and u is smooth (for example, bounded and Lipschitz continuous), then the integral in (2.1) is in fact not singular near x. We also recall that using the Fourier transform, $(-\Delta)^s$ can be also defined as a pseudo-differential operator with symbol $|\xi|^{2s}$. Let $G \subset \mathbb{R}^N$ be an arbitrary bounded open set. Consider next the space

$$\mathcal{D}(G) := \Big\{u : G \to \mathbb{R}\ \text{ measurable}, u \in C^\infty(G),\ \text{supp}[u]\ \text{ is compact in }\ G\Big\}.$$

Let now $u \in \mathcal{D}(G)$. Since $u = 0$ on $\mathbb{R}^N\backslash G$, a simple calculation gives

$$(-\Delta)^s_G u = (-\Delta)^s u(x) - V_G(x)u,$$

where the potential V_G is given by

$$V_G(x) := C_{N,s}\int_{\mathbb{R}^N\backslash G}|x-y|^{-N-2s}dy, \quad x \in G. \tag{2.2}$$

More precisely, we have

$$(-\Delta)^s u = (-\Delta)^s_G u + V_G(x)u, \quad \text{for all }\ u \in \mathcal{D}(G). \tag{2.3}$$

Based on (2.3), we then view the fractional Laplacian $(-\Delta)^s$ with domain $\mathcal{D}(G)$ as a perturbation of the regional fractional operator $(-\Delta)^s_G$ with the non-negative potential V_G. This allows us to observe that $(-\Delta)^s_G$ describes a particle jumping from one point $x \in G$ to another $y \in G$ with intensity proportional to $\left|x-y\right|^{-N-2s}$.

Remark 2.1. We mention that when starting with a function defined only on G, then a relation like the one in (2.3) makes only sense if one extends the function by 0 on $\mathbb{R}^N\backslash G$. If one has a non-zero extension $\widetilde{u}$ on $\mathbb{R}^N\backslash G$, then for such a function there is no relation between $(-\Delta)^s\widetilde{u}$ and $(-\Delta)^s_G u$. In our situation, since we are considering fractional Neumann type boundary conditions, the extension by zero of the involved functions on the complement of the domain is not allowed (see, e.g. [51] and the references therein for more details on this topic). In addition, although our transmission problem involves nonlocal operators in (1.8) and (1.9) the transmission condition in (1.10) is purely “local” in nature since it is satisfied only over Σ. At a first look, it is not clear if one replaces the regional fractional Laplacian by the fractional Laplace operator as in [31] (cf. also [17, Introduction] for a discussion in the context of dynamic fractional boundary conditions), that one can still obtain a local boundary condition. We refer to Section 5 below for a discussion on this topic.

Let now $G \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary ∂G. Let

$$W^{1,2}(G) := \left\{u \in L^2(G) : \int_G |\nabla u|^2\,dx < \infty\right\}$$

be the classical first order Sobolev space. For $s \in (0,1)$, we denote by

$$W^{s,2}(G) := \left\{u \in L^2(G) : \int_G\int_G \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy < \infty\right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,2}(G)} := \left(\int_G |u|^2 dx + \int_G\int_G \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}.$$

Let $0 < s \leq 1$. By [8, Theorem 6.7], for all q satisfying

$$q \in [1, 2^\star] \text{ with } 2^\star := \frac{2N}{N-2s} \text{ if } N > 2s \text{ and } q \in [1,\infty) \text{ if } N = 2s,$$

the continuous embedding

$$W^{s,2}(G) \hookrightarrow L^q(G) \tag{2.4}$$

holds. That is, there exists a constant $C > 0$ such that for every $u \in W^{s,2}(G)$,

$$\|u\|_{L^q(G)} \leq C\|u\|_{W^{s,2}(G)}.$$

It also follows from (2.4) that for every $q \in [1, 2^\star)$, the embedding $W^{s,2}(G) \hookrightarrow L^q(G)$ is compact (see e.g. [8, Section 7]).

If $N < 2s$, then one has the continuous embedding

$$W^{s,2}(G) \hookrightarrow C^{0,s-\frac{N}{2}}(\overline{G}). \tag{2.5}$$

We also let $W_0^{s,2}(G) := \overline{\mathcal{D}(G)}^{W^{s,2}(G)}$, that is, the closure of $\mathcal{D}(G)$ in $W^{s,2}(G)$.

Remark 2.2. We mention that in our situation since we shall consider $1/2 < s < 1$, we have that $2^\star = \frac{2N}{N-2s}$ for $N \geq 2 > 2s$. If $N = 1$, then $N = 1 < 2s$ and hence we have the embedding (2.5). We also notice that, since we have assumed that G has a Lipschitz continuous boundary, then by [3] (see also [50, Theorem 4.8] for a more general version), $W_0^{s,2}(G) = W^{s,2}(G)$ for every $0 < s \leq 1/2$. Finally we recall that $W_0^{s,2}(G)$ is not to be confused with the space $W_0^{s,2}(\overline{G}) = \{u \in W^{s,2}(\mathbb{R}^N),\ u = 0 \text{ on } \mathbb{R}^N\backslash G\}$. They coincide if $\frac{1}{2} < s < 1$ but they are different if $s = 1/2$. The latter is the right space to define the fractional Laplacian with Dirichlet boundary condition.

Next, let $F \subset \mathbb{R}^N$ be a Lipschitz hypersurface of dimension $N-1$ and let σ be the restriction to F of the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. We also introduce the fractional order Sobolev space on F for $0 < s < 1$ as the Banach space

$$W^{s,2}(F) = \left\{u \in L^2(F) : \int_F\int_F \frac{|u(x)-u(y)|^2}{|x-y|^{N-1+2s}}d\sigma_x d\sigma_y < \infty\right\}$$

endowed with the norm given by

$$\|u\|_{W^{s,2}(F)} := \left(\int_F |u|^2 d\sigma + \int_F\int_F \frac{|u(x)-u(y)|^2}{|x-y|^{N-1+2s}} d\sigma_x d\sigma_y\right)^{\frac{1}{2}}.$$

By [6, Theorem 11.1], if $1/2 < s \le 1$, then there exists a constant $C > 0$ such that for every $u \in W^{s,2}(G)$,

$$\|u\|_{L^r(\partial G)} \le C\|u\|_{W^{s,2}(G)}, \tag{2.6}$$

for all r satisfying

$$r \in [1, 2_\star] \text{ with } 2_\star := \frac{2(N-1)}{N-2s} \text{ if } N > 2s \text{ and } r \in [1, \infty) \text{ if } N = 2s.$$

In fact if $F = \partial G$ for some open set $G \subset \mathbb{R}^N$ (hence G has a Lipschitz continuous boundary), then one has the continuous embedding

$$W^{s,2}(G) \hookrightarrow W^{s-\frac{1}{2},2}(\partial G) \hookrightarrow L^r(\partial G). \tag{2.7}$$

It also follows from (2.6) that for every $r \in [1, 2_\star)$, the embedding $W^{s,2}(G) \hookrightarrow L^r(\partial G)$ is compact.

Next, let Ω, Ω_+, Ω_- and Σ be as in Figure 1. Throughout the remainder of the paper for a function $u \in L^2(\Omega\backslash\Sigma)$ we let $u_+ := u|_{\Omega_+}$ and $u_- := u|_{\Omega_-}$. Let $1/2 < \beta \le \alpha < 1$ and let w be a fixed smooth function defined in Σ. We define the fractional order Sobolev space

$$\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) := \Big\{u \in L^2(\Omega\backslash\Sigma) : u_- \in W^{\alpha,2}(\Omega_-),\ u_+ \in W^{\beta,2}(\Omega_+) \text{ and } u_+ = wu_- \text{ on } \Sigma\Big\}, \tag{2.8}$$

and we endow it with the norm defined by

$$\|u\|^2_{\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)} = \int_{\Omega\backslash\Sigma} |u|^2 dx + \int_{\Omega_-}\int_{\Omega_-} \frac{|u_-(x)-u_-(y)|^2}{|x-y|^{N+2\alpha}} dxdy + \int_{\Omega_+}\int_{\Omega_+} \frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2\beta}} dxdy.$$

Remark 2.3. We notice that if $0 < \beta \le 1/2$, then it follows from Remark 2.2 that $W^{\beta,2}(\Omega_+) = W_0^{\beta,2}(\Omega_+)$. Although the boundary trace of u_+ does not exist in the case $\beta \in (0, 1/2]$ (see also [32, Section 4.1]), we can say that $u_+ = 0$ (in the sense of capacity, see [3, Corollary 2.8] or [50]) on Σ and according to (2.8) we must then have $w = 0$ on Σ (in the sense of capacity) unless $0 < \alpha \le 1/2$, in which case $u_- = 0$ on Σ (in the sense of capacity), as well. This is precisely the reason we have assumed $1/2 < \beta \le \alpha < 1$. Let us now also define the jump $[u]_\Sigma$ of u across the interface Σ, $[u]_\Sigma := u_+|_\Sigma - u_-|_\Sigma$, provided that the latter are well defined as traces. If one chooses $\alpha = \beta$ and $w \equiv 1$, our transmission conditions in (1.10) become the familiar ones. Indeed, take $w \equiv 1$ in (2.8) and observe that one can rewrite the boundary value problem (1.8)–(1.11), assuming of course $f = f_\pm$, as follows:

$$\begin{cases} \partial_t u + (-\Delta)^\alpha_\Omega u + f(u) = 0 & \text{in } J \times \Omega, \\ [u]_\Sigma = 0,\ C_\alpha \left[\mathcal{N}^{2-2\alpha} u\right]_\Sigma + \mathfrak{b} u = 0, & \text{on } J \times \Sigma, \\ \mathcal{N}^{2-2\alpha} u = 0, & \text{on } J \times \Gamma. \end{cases}$$

However, in general we have the following.

Proposition 2.4. *Let $w \in W^{\theta,2}(\Sigma)$ with $\theta \geq \beta - 1/2$ such that $\theta + \alpha - \beta > (N-1)/2$. Then we have that $wu_- \in W^{\beta-\frac{1}{2},2}(\Sigma)$ for every $u_- \in W^{\alpha-\frac{1}{2},2}(\Sigma)$.*

Proof. The claim follows from the application of Lemma F (see Appendix) with $w_1 := w$, $w_2 := u_-$, $s_1 := \theta$, $s_2 := \alpha - 1/2$ and $s := \beta - 1/2$. □

Since we have assumed that $N \geq 2$ and that $1 < 2\beta \leq 2\alpha < 2$, and since both Ω_+ and Ω_- have Lipschitz continuous boundary, it follows from (2.4) that for $N \geq 2$ we have the following continuous embeddings:

$$\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow L^{\frac{2N}{N-2\alpha}}(\Omega_+) \text{ and } \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow L^{\frac{2N}{N-2\beta}}(\Omega_-).$$

This implies the continuous embedding (since $L^{\frac{2N}{N-2\alpha}}(\Omega_+) \hookrightarrow L^{\frac{2N}{N-2\beta}}(\Omega_+)$)

$$\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow L^{\frac{2N}{N-2\beta}}(\Omega\backslash\Sigma). \tag{2.9}$$

In addition, using (2.6) and (2.7), we have the following continuous embeddings:

$$\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow W^{\alpha,2}(\Omega_+) \hookrightarrow W^{\alpha-\frac{1}{2},2}(\Sigma) \hookrightarrow L^{\frac{2(N-1)}{N-2\alpha}}(\Sigma) \tag{2.10}$$

and

$$\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow W^{\beta,2}(\Omega_-) \hookrightarrow W^{\beta-\frac{1}{2},2}(\Gamma\cup\Sigma) \hookrightarrow L^{\frac{2(N-1)}{N-2\beta}}(\Gamma\cup\Sigma). \tag{2.11}$$

We notice the following result.

Lemma 2.5. *We have that*

$$\|u\|_*^2 := \int_\Sigma |u_-|^2 d\sigma + \int_{\Omega_-}\int_{\Omega_-} \frac{|u_-(x)-u_-(y)|^2}{|x-y|^{N+2\alpha}}dxdy + \int_{\Omega_+}\int_{\Omega_+} \frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2\beta}}dxdy, \tag{2.12}$$

defines an equivalent norm on $\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$.

Proof. For $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$, we let $g(u) := \int_\Sigma |u_-|^2 d\sigma$. Now let $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$ and assume that $\|u\|_* = 0$. Then $u_+ = C_+$ on Ω_+ and $u_- = C_-$ on Ω_- for some constants C_+ and C_-. Since $g(u) = \|u\|^2_{L^2(\Sigma)} = 0$, we have that $u_- = 0$ σ -a.e. on Σ. It follows from the uniqueness of the trace that $C_- = 0$. Since $u_+ = wu_-$ σ-a.e. on Σ, we have that $u_+ = 0$ σ-a.e. on Σ and this also implies that $C_+ = 0$. Thus $u = 0$ a.e. on $\Omega\backslash\Sigma$. Now using (2.10) and proceeding as the proof of [39, Theorem 1.1.15, p. 27] we get the claim. □

For more information on fractional order Sobolev spaces we refer to [6, 8, 21, 30, 37, 50] and their references.

2.2. The fractional normal derivative

In this (sub)section, we introduce the fractional normal derivative mentioned in the introduction. We start with smooth open sets. Let $G \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$

with boundary ∂G. The following definition is taken from [22, Definition 2.1] (see also [23, Definition 7.1] for the one-dimensional case).

Definition 2.6. For $u \in C^1(G)$, $z \in \partial G$ and $1/2 < s < 1$, we define the boundary operator $\mathcal{N}^{2-2s}$ by

$$\mathcal{N}^{2-2s}u(z) = \lim_{t\downarrow 0} \frac{du(z + t\vec{\nu}(z))}{dt} t^{2-2s}, \tag{2.13}$$

whenever the limit exists, where $\vec{\nu}(z)$ denotes the outer normal vector to G at the point z.

It is easy to see that if $u \in C(\overline{G})$, then (2.13) is equivalent to

$$\mathcal{N}^{2-2s}u(z) = \lim_{t\downarrow 0} \frac{u(z + t\vec{\nu}(z)) - u(z)}{t^{2s-1}},$$

so that if $s = 1$, it coincides with $\partial_\nu u$, that is, the normal derivative of u in direction of the outer normal vector $\vec{\nu}$.

Next, let $1/2 < s < 1$, $\rho(x) := \text{dist}(x, \partial G)$, $x \in G$ and define the space

$$C^2_{2s}(\overline{G}) := \{u : \ u(x) = f(x)(\rho(x))^{2s-1} + g(x), \ \forall\, x \in G, \ \text{ for some } f, g \in C^2(\overline{G})\},$$

and we always assume that $u \in C^2_{2s}(\overline{G})$ is defined on $\overline{G}$ by continuous extension.

The following fractional Green type formula for the regional fractional Laplace operator has been obtained in [22, Theorem 3.3] (see also [50, Theorem 5.7] for a weak form and a more general version). For $t \geq 0$, we set $t \vee 1 := \max\{1, t\}$ and $t \wedge 1 := \min\{1, t\}$.

Theorem 2.7. *Let $1/2 < s < 1$. Then, for every $u \in C^2_{2s}(\overline{G})$ and $v \in W^{s,2}(G)$, we have*

$$\int_G v(-\Delta)^s_G u\, dx = \frac{C_{N,s}}{2} \int_G \int_G \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy - C_s \int_{\partial G} v\mathcal{N}^{2-2s} u d\sigma,$$

where C_s is an explicit constant depending only on s and is given by

$$C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{|t-1|^{1-2s} - (t \vee 1)^{1-2s}}{t^{2-2s}} dt. \tag{2.14}$$

We adopt the following definition.

Definition 2.8. For $1/2 < s < 1$ and $u \in C^2_{2s}(\overline{G})$, we call $C_s\mathcal{N}^{2-2s}u$ *the strong fractional normal derivative* of u in direction of the outer normal vector.

Next, we introduce a weak formulation on non-smooth domains of a fractional normal derivative.

Definition 2.9. Let $G \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary ∂G and $1/2 < s < 1$.

(a) Let $u \in W^{s,2}(G)$. We say that $(-\Delta)^s_G u \in L^2(G)$ if there exists $f \in L^2(G)$ such that

$$\frac{C_{N,s}}{2} \int_G \int_G \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} dxdy = \int_G fv dx$$

for all $v \in \mathcal{D}(G)$. In that case we write $(-\Delta)^s_G u = f$.

(b) Let $u \in W^{s,2}(G)$ such that $(-\Delta)^s_G u \in L^2(G)$. We say that u has a weak fractional normal derivative in $L^2(\partial G)$ if there exists $g \in L^2(\partial G)$ such that

$$\int_G v(-\Delta)^s_G u\, dx = \frac{C_{N,s}}{2}\int_G\int_G \frac{(v(x)-v(y))(u(x)-u(y))}{|x-y|^{N+2s}}dxdy - \int_{\partial G} gv d\sigma \quad (2.15)$$

for all $v \in W^{s,2}(G) \cap C(\overline{G})$. In that case, g is uniquely determined by (2.15), we write $C_s\mathcal{N}^{2-2s}u = g$ and call g the *weak fractional normal derivative* of u.

Remark 2.10. It follows from Definition 2.9 that the *Green's type formula*

$$\int_G v(-\Delta)^s_G u\, dx = \frac{C_{N,s}}{2}\int_G\int_G \frac{(v(x)-v(y))(u(x)-u(y))}{|x-y|^{N+2s}}dxdy - C_s\int_{\partial G} v\mathcal{N}^{2-2s}u d\sigma, \quad (2.16)$$

holds for all $v \in W^{s,2}(G)$ whenever $u \in W^{s,2}(G)$, $(-\Delta)^s_G u \in L^2(G)$ and $\mathcal{N}^{2-2s}u$ exists in $L^2(\partial G)$.

We also notice that if G is a bounded open set of class $C^{1,1}$, then on $C^2_{2s}(\overline{G})$, weak and strong fractional normal derivatives coincide. Moreover by [22, 51], we have that for every $u \in C^2_{2s}(\overline{G})$, $\mathcal{N}^{2-2s}u \in L^\infty(\partial G)$ and $(-\Delta)^s_G u \in L^p(G)$ for every $p \in [1,\infty)$. In fact the weak fractional normal derivative, of a function u satisfying $u \in W^{s,2}(G)$, $(-\Delta)^s_G u \in L^2(G)$ and $\mathcal{N}^{2-2s}u$ exists in $L^2(\partial G)$, is obtained by approximating such a function by a sequence of functions in $C^2_{2s}(\overline{G})$ where the strong fractional normal derivative exists, and then pass to the limit. For more details on this topic we refer to [51].

2.3. The linear parabolic problem

Let $\Omega, \Omega_+, \Omega_-$ and Σ be as above (see Figure 1) and let $\mathfrak{b} \in L^\infty(\Sigma)$ be a nonnegative function. Recall that $1/2 < \beta \le \alpha < 1$. We define the bilinear symmetric form $\mathcal{E}_{\alpha,\beta}$ with domain $D(\mathcal{E}_{\alpha,\beta}) := \mathbb{W}^{(\alpha,\beta),2}(\Omega\setminus\Sigma)$ by

$$\mathcal{E}_{\alpha,\beta}(u,v) = \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy + \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy + \int_\Sigma \mathfrak{b}u_-v_- d\sigma. \quad (2.17)$$

We have the following result.

Proposition 2.11. *The form $\mathcal{E}_{\alpha,\beta}$ is a Dirichlet form on $L^2(\Omega\setminus\Sigma)$ and $D(\mathcal{E}_{\alpha,\beta})$ is dense in $L^2(\Omega\setminus\Sigma)$.*

Proof. First we claim that $\mathcal{E}_{\alpha,\beta}$ is closed. Indeed, let $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{E}_{\alpha,\beta})$ be such that

$$\lim_{n,m\to\infty}\left(\mathcal{E}_{\alpha,\beta}(u_n-u_m,u_n-u_m) + \|u_n-u_m\|^2_{L^2(\Omega\setminus\Sigma)}\right) = 0. \quad (2.18)$$

It follows from (2.18) that $\{u_{n-}\}_{n\in\mathbb{N}} := \{u_n|_{\Omega_-}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{\alpha,2}(\Omega_-)$. Hence, there exists a $u_- \in W^{\alpha,2}(\Omega_-)$ such that u_{n-} converges strongly to u_- in $W^{\alpha,2}(\Omega_-)$.

Since $W^{\alpha,2}(\Omega_-) \hookrightarrow L^2(\Sigma)$, we also have that $u_{n-}|_\Sigma$ converges strongly to $u_-|_\Sigma$ in $L^2(\Sigma)$ and hence σ-a.e. after a sub-sequence if necessary. Similarly, we have that $\{u_{n+}\}_{n\in\mathbb{N}} := \{u_n|_{\Omega_+}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{\beta,2}(\Omega_+)$. Then there exists a $u_+ \in W^{\beta,2}(\Omega_-)$ such that u_{n+} converges strongly to u_+ in $W^{\beta,2}(\Omega_+)$. We also have that $u_{n+}|_\Sigma$ converges strongly to $u_+|_\Sigma$ in $L^2(\Sigma)$ and σ-a.e. after a sub-sequence if necessary. Since by hypothesis $u_{n+} = wu_{n-}$ σ-a.e. on Σ for every $n \in \mathbb{N}$, it follows from the above convergences that $u_+ = wu_-$ σ-a.e. on Σ. Define u on $\Omega\backslash\Sigma$ such that $u := u_+$ on Ω_+ and $u := u_-$ on Ω_-. Then $u \in D(\mathcal{E}_{\alpha,\beta})$ and a simple calculation shows that $\lim_{n\to\infty} \mathcal{E}_{\alpha,\beta}(u_n - u, u_n - u) = 0$. This proves the claim.

Next, we claim that $\mathcal{E}_{\alpha,\beta}$ is Markovian. Indeed, let $0 \le u \in D(\mathcal{E}_{\alpha,\beta})$. Then proceeding as in [50, Lemma 2.7] we get that $u \wedge 1 \in D(\mathcal{E}_{\alpha,\beta})$ and $\mathcal{E}_{\alpha,\beta}(u \wedge 1, u \wedge 1) \le \mathcal{E}_{\alpha,\beta}(u, u)$. By [13, p.5], this implies that $\mathcal{E}_{\alpha,\beta}$ is Markovian. We have shown that $\mathcal{E}_{\alpha,\beta}$ is a Dirichlet form. Since $\mathcal{D}(\Omega\backslash\Sigma) \subset D(\mathcal{E}_{\alpha,\beta})$ and is dense in $L^2(\Omega\backslash\Sigma)$, we have that $D(\mathcal{E}_{\alpha,\beta})$ is dense in $L^2(\Omega\backslash\Sigma)$ and the proof is finished. $\square$

Let $A_{\alpha,\beta}$ be the self-adjoint operator on $L^2(\Omega\backslash\Sigma)$ associated with $\mathcal{E}_{\alpha,\beta}$ in the sense that

$$\begin{cases} D(A_{\alpha,\beta}) := \Big\{u \in D(\mathcal{E}_{\alpha,\beta}),\ \exists\, f \in L^2(\Omega\backslash\Sigma),\ \mathcal{E}_{\alpha,\beta}(u, v) = (f, v)_{L^2(\Omega\backslash\Sigma)}\ \forall\, v \in D(\mathcal{E}_{\alpha,\beta})\Big\} \\ A_{\alpha,\beta} u = f. \end{cases} \tag{2.19}$$

We have the following characterization of the operator $A_{\alpha,\beta}$.

Proposition 2.12. *The operator $A_{\alpha,\beta}$ is given by*

$$D(A_{\alpha,\beta}) = \Big\{u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma),\ (-\Delta)^\alpha_{\Omega_-} u_- \in L^2(\Omega_-),\ (-\Delta)^\beta_{\Omega_+} u_+ \in L^2(\Omega_+),$$
$$\mathcal{N}^{2-2\alpha} u_- = 0 \ \text{ on } \ \Gamma,\ \mathcal{N}_w(u_+, u_-) + \mathfrak{b} u_- = 0 \ \text{ on } \ \Sigma\Big\} \tag{2.20}$$

and, for $u \in D\left(A_{\alpha,\beta}\right)$, we have that

$$A_{\alpha,\beta} u = (-\Delta)^\beta_{\Omega_+} u_+ \text{ on } \Omega_+, \text{ and } A_{\alpha,\beta} u = (-\Delta)^\alpha_{\Omega_-} u_- \text{ on } \Omega_-. \tag{2.21}$$

In addition, $A_{\alpha,\beta}$ has a compact resolvent, and hence, has a discrete spectrum. The spectrum of $A_{\alpha,\beta}$ is an increasing nonnegative sequence of real numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \to \infty$. If $\mathfrak{b}$ satisfies

$$\mathfrak{b}(x) \ge \mathfrak{b}_0 > 0 \quad \text{on} \quad \Sigma, \tag{2.22}$$

then $\lambda_1 > 0$. If $\mathfrak{b} = 0$ σ-a.e. on Σ, then $\lambda_1 = 0$.

Proof. Let D be given by the right hand side of (2.19) and $D(A_{\alpha,\beta})$ the right hand side of (2.20). Let $u \in D(A_{\alpha,\beta})$ and then set $f_- := (-\Delta)^\alpha_{\Omega_-} u_-$ in Ω_-, and $f_+ := (-\Delta)^\beta_{\Omega_+} u_+$ in Ω_+. Then by hypothesis we have that $f \in L^2(\Omega\backslash\Sigma)$, $\mathcal{N}^{2-2\alpha} u_-$ exists on $\Gamma \cup \Sigma$ and $\mathcal{N}^{2-2\beta} u_+$ exists on Σ. Let $v \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$. Then using the integration by parts formula (2.16) and the fact that the outer normal vectors to Ω_+ and Ω_- at Σ have opposite sign, we get that

$$\int_{\Omega\backslash\Sigma} f v dx = \int_{\Omega_+} f_+ v_+ dx + \int_{\Omega_-} f_- v_- dx = \int_{\Omega_+} v_+ (-\Delta)^\beta_{\Omega_+} u_+ dx + \int_{\Omega_-} v (-\Delta)^\alpha_{\Omega_-} u_- dx$$

$$= \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy - C_\beta\int_\Sigma v_+\mathcal{N}^{2-2\beta}u_+d\sigma$$
$$+ \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy$$
$$+ C_\alpha\int_\Sigma v_-\mathcal{N}^{2-2\alpha}u_-d\sigma - C_\alpha\int_\Gamma v_-\mathcal{N}^{2-2\alpha}u_-d\sigma. \tag{2.23}$$

Since $v_+ = wv_-$ and $\mathcal{N}^{2-2\alpha}u_- = 0$ on Γ, it follows from (2.23) that

$$\int_{\Omega\setminus\Sigma} fvdx = \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy$$
$$+ \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy$$
$$- \int_\Sigma\left(C_\beta w\mathcal{N}^{2-2\beta}u_+ - C_\alpha\mathcal{N}^{2-2\alpha}u_-\right)v_-d\sigma. \tag{2.24}$$

Finally using $C_\beta w\mathcal{N}^{2-2\beta}u_+ - C_\alpha\mathcal{N}^{2-2\alpha}u_- = -\mathfrak{b}u_-$ on Σ, we get from (2.24) that

$$\int_{\Omega\setminus\Sigma} fvdx = \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy$$
$$+ \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy$$
$$+ \int_\Sigma \mathfrak{b}u_-v_-d\sigma = \mathcal{E}_{\alpha,\beta}(u,v).$$

Hence, $u \in D$ and $A_{\alpha,\beta}u = f$. We have shown that $D(A_{\alpha,\beta}) \subset D$.

Conversely, let $u \in D$. Then by definition, there exists a $f \in L^2(\Omega\setminus\Sigma)$ such that the equality

$$\int_{\Omega\setminus\Sigma} fvdx = \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy$$
$$+ \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy$$
$$+ \int_\Sigma \mathfrak{b}u_-v_-d\sigma \tag{2.25}$$

holds for every $v \in D(\mathcal{E}_{\alpha,\beta})$. In particular we deduce from (2.25) that for every $v \in \mathcal{D}(\Omega_+) \subset D(\mathcal{E}_{\alpha,\beta})$,

$$\int_{\Omega_+} f_+v_+dx = \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+}\frac{(u_+(x)-u_+(y))(v_+(x)-v_+(y))}{|x-y|^{N+2\beta}}dxdy.$$

By Definition 2.9-(a), the preceding identity implies that $(-\Delta)^\beta_{\Omega_+}u_+ \in L^2(\Omega_+)$ and $(-\Delta)^\beta_{\Omega_+}u_+ = f_+$ on Ω_+. Similarly, it follows from (2.25) that for every $v \in \mathcal{D}(\Omega_-) \subset D(\mathcal{E}_{\alpha,\beta})$,

$$\int_{\Omega_-} f_-v_-dx = \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-}\frac{(u_-(x)-u_-(y))(v_-(x)-v_-(y))}{|x-y|^{N+2\alpha}}dxdy.$$

Once again by Definition 2.9-(a), this also implies that $(-\Delta)^{\alpha}_{\Omega_-}u_- \in L^2(\Omega_-)$ and $(-\Delta)^{\alpha}_{\Omega_-}u_- = f_-$ on Ω_-. Next, by Definition 2.9-(b), we also get from (2.25) that $\mathcal{N}^{2-2\alpha}u_- = 0$ on Γ and

$$\int_\Sigma \Big(C_\beta v_+ \mathcal{N}^{2-2\beta}u_+ - C_\alpha v_- \mathcal{N}^{2-2\alpha}u_- + \mathfrak{b}u_-v_-\Big)d\sigma = 0 \tag{2.26}$$

for every $v \in D(\mathcal{E}_{\alpha,\beta})$. Since $v_+ = wv_-$, it follows from (2.26) that

$$\int_\Sigma \Big(C_\beta w\mathcal{N}^{2-2\beta}u_+ - C_\alpha \mathcal{N}^{2-2\alpha}u_- + \mathfrak{b}u_-\Big)v_- d\sigma = \int_\Sigma \Big(\mathcal{N}_w(u_+,u_-) + \mathfrak{b}u_-\Big)v_- d\sigma = 0,$$

for every $v \in D(\mathcal{E}_{\alpha,\beta})$. This implies that $\mathcal{N}_w(u_+,u_-) + \mathfrak{b}u_- = 0$ on Σ. We have shown that $u \in D(A_{\alpha,\beta})$ and $A_{\alpha,\beta}u$ is given by (2.21).

Finally, (2.9) implies that the embedding $D(\mathcal{E}_{\alpha,\beta}) \hookrightarrow L^2(\Omega\backslash\Sigma)$ is compact and this shows that the operator $A_{\alpha,\beta}$ has a compact resolvent. Since $A_{\alpha,\beta}$ is a positive self-adjoint operator with compact resolvent, then it has a discrete spectrum formed of eigenvalues satisfying

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \text{ and } \lim_{k\to\infty} \lambda_k = +\infty.$$

If $\mathfrak{b}$ satisfies (2.22), then it follows from (2.12) that $(\mathcal{E}_{\alpha,\beta}(u,u))^{1/2}$ defines an equivalent norm on $D(\mathcal{E}_{\alpha,\beta})$ and this implies that $\lambda_1 > 0$. Finally, if $\mathfrak{b} = 0$, then the constant function $1 \in D(A_{\alpha,\beta})$ and $A_{\alpha,\beta}1 = 0$. Hence, $\lambda_1 = 0$. The proof of the proposition is finished. □

Definition 2.13. Let X be a locally compact metric space and $\mathfrak{m}$ a Radon measure on X. Let $(T(t))_{t\ge0}$ be a strongly continuous semigroup on $L^2(X,\mathfrak{m})$.

(a) We say that the semigroup is *positive preserving* if

$$T(t)u \ge 0, \text{ for all } t \ge 0 \text{ whenever } u \in L^2(X,\mathfrak{m}),\ u \ge 0 \text{ a.e.}$$

(b) We say that the semigroup is L^∞-contractive if

$$\|T(t)u\|_{L^\infty(X,\mathfrak{m})} \le \|u\|_{L^\infty(X,\mathfrak{m})}, \text{ for all } u \in L^\infty(X,\mathfrak{m}) \cap L^2(X,\mathfrak{m}) \text{ and } t \ge 0.$$

A positive preserving and L^∞-contractive semigroup is called *submarkovian*.

We have the following result of generation of semigroup. We note that by $u \ge 0$ a.e. on $\Omega\backslash\Sigma$, we mean that $u_+ \ge 0$ (a.e.) on Ω_+ and $u_- \ge 0$ (a.e.) on Ω_-.

Proposition 2.14. *Let $A_{\alpha,\beta}$ be the self-adjoint operator on $L^2(\Omega\backslash\Sigma)$ defined in (2.20) and (2.21). Then $-A_{\alpha,\beta}$ generates a strongly continuous submarkovian semigroup $(e^{-tA_{\alpha,\beta}})_{t\ge0}$ on $L^2(\Omega\backslash\Sigma)$ which is also compact.*

Proof. We have shown in Proposition 2.11 that $\mathcal{E}_{\alpha,\beta}$ is a Dirichlet form on $L^2(\Omega\backslash\Sigma)$ and that $D(\mathcal{E}_{\alpha,\beta})$ is dense in $L^2(\Omega\backslash\Sigma)$. Hence, by [13, Theorem 1.4.1], $-A_{\alpha,\beta}$ generates a strongly continuous semigroup $(e^{-tA_{\alpha,\beta}})_{t\ge0}$ on $L^2(\Omega\backslash\Sigma)$ which is L^∞-contractive. Next, let $u \in D(\mathcal{E}_{\alpha,\beta})$. Then proceeding as in [50, Lemma 2.6] we get that $u^+ := u \vee 0$, $u^- := 0 \wedge (-u) \in D(\mathcal{E}_{\alpha,\beta})$ and a simple calculation gives

$$\mathcal{E}_{\alpha,\beta}(u^+,u^-) = -\frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+} \frac{u_+^+(x)u_+^-(y) + u_+^+(y)u_+^-(x)}{|x-y|^{N+2\beta}}dxdy$$
$$- \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-} \frac{u_-^+(x)u_-^-(y) + u_-^+(y)u_-^-(x)}{|x-y|^{N+2\alpha}}dxdy \le 0.$$

Hence by [7, Theorem 1.3.2], the semigroup is also positive preserving. Since $A_{\alpha,\beta}$ has a compact resolvent, it follows that the semigroup is also compact. The proof is finished. □

Remark 2.15. Since the semigroup $(e^{-tA_{\alpha,\beta}})_{t\geq 0}$ is submarkovian, it follows from [7, Theorem 1.4.1] that it can then be extended to contraction semigroups $S_{\alpha,\beta,p}(t) := e^{-tA_{\alpha,\beta,p}}$ on $L^p(\Omega\backslash\Sigma)$ for every $p \in [1,\infty]$, and each semigroup is strongly continuous if $p \in [1,\infty)$ and bounded analytic if $p \in (1,\infty)$. Denote by $A_{\alpha,\beta,p}$ the generator of the semigroup on $L^p(\Omega\backslash\Sigma)$ so that $A_{\alpha,\beta,2} = A_{\alpha,\beta}$. The semigroups are also consistent in the sense that for all $t \geq 0$,

$$e^{-tA_{\alpha,\beta,p}}f = e^{-tA_{\alpha,\beta,q}}f \quad \text{if } f \in L^p(\Omega\backslash\Sigma) \cap L^q(\Omega\backslash\Sigma).$$

They are self-adjoint in the sense that if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for all $t \geq 0$,

$$\left(e^{-tA_{\alpha,\beta,p}}\right)^\star = e^{-tA_{\alpha,\beta,q}} \quad \text{and} \quad A^\star_{\alpha,\beta,p} = A_{\alpha,\beta,q}.$$

The operator $A_{\alpha,\beta,\infty}$ is defined as

$$\left(\lambda + A_{\alpha,\beta,\infty}\right)^{-1} = \left[\left(\lambda + A_{\alpha,\beta,1}\right)^{-1}\right]^\star$$

for all $\lambda > 0$ and notice that its domain is not dense in $L^\infty(\Omega\backslash\Sigma)$. Since $L^p(\Omega\backslash\Sigma) \subseteq L^1(\Omega\backslash\Sigma)$ and $S_{\alpha,\beta,p}(t) \subseteq S_{\alpha,\beta,1}(t)$ for all $p \in [1,\infty]$, we can drop the index p and merely write $S_{\alpha,\beta}$ for the semigroup for the sake of notational simplicity.

Next, we give and prove some crucial ultracontractivity estimates for the semigroup.

Theorem 2.16. *The semigroup $(e^{-tA_{\alpha,\beta}})_{t\geq 0}$ is ultracontractive in the sense that it maps $L^2(\Omega\backslash\Sigma)$ into $L^\infty(\Omega\backslash\Sigma)$. More precisely we have the following.*

(a) If $\mathfrak{b}$ satisfies (2.22), then for every $1 \leq q \leq p \leq \infty$, there exists a constant $C > 0$ such that for every $f \in L^q(\Omega\backslash\Sigma)$ and $t > 0$,

$$\|e^{-tA_{\alpha,\beta}}f\|_{L^p(\Omega\backslash\Sigma)} \leq Ce^{-\lambda_1\left(\frac{1}{q}-\frac{1}{p}\right)t}t^{-\frac{N}{2\beta}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\Omega\backslash\Sigma)}, \tag{2.27}$$

where we recall that $\lambda_1 > 0$ is the first eigenvalue of $A_{\alpha,\beta}$.

(b) If $\mathfrak{b} = 0$, then there exists a constant $C > 0$ such that for every $f \in L^q(\Omega\backslash\Sigma)$ and $t > 0$,

$$\|e^{-tA_{\alpha,\beta}}f\|_{L^p(\Omega\backslash\Sigma)} \leq C(t \wedge 1)^{-\frac{N}{2\beta}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\Omega\backslash\Sigma)}. \tag{2.28}$$

In addition, we have that each semigroup on $L^p(\Omega\backslash\Sigma)$ is compact for every $p \in [1,\infty]$ and if u_k is an eigenfunction of $A_{\alpha,\beta}$ associated with the eigenvalue λ_k, then $u_k \in D\left(A_{\alpha,\beta}\right) \cap L^\infty(\Omega\backslash\Sigma)$.

Proof. Recall that there exist consistent semigroups of contractions on $L^p(\Omega\backslash\Sigma)$, $p \in [1,\infty]$, that is,

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^p(\Omega\backslash\Sigma))} \leq 1, \quad \text{for all } t \geq 0. \tag{2.29}$$

Since the semigroup is also analytic on $L^2(\Omega\backslash\Sigma)$, we have that $e^{-tA_{\alpha,\beta}}\varphi \in D(A_{\alpha,\beta}) \subset \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$ for every $\varphi \in L^2(\Omega\backslash\Sigma)$ and $t > 0$. Moreover, there exists a constant $C > 0$ such that for every $\varphi \in L^2(\Omega\backslash\Sigma)$ and $t > 0$,

$$\|A_{\alpha,\beta}e^{-tA_{\alpha,\beta}}\varphi\|_{L^2(\Omega\backslash\Sigma)} \leq \frac{C}{t}\|\varphi\|_{L^2(\Omega\backslash\Sigma)}. \tag{2.30}$$

Recall that

$$2^\star := \frac{2N}{N-2\beta} = \frac{2\mu}{\mu-2}, \quad \mu := \frac{N}{\beta}.$$

(a) If $\mathfrak{b}$ satisfies (2.22), then it follows from (2.9) and (2.12) that there exists a constant $C > 0$ such that for every $u \in D(\mathcal{E}_{\alpha,\beta})$,

$$\|u\|^2_{L^{2^\star}(\Omega\setminus\Sigma)} \le C\mathcal{E}_{\alpha,\beta}(u,u). \tag{2.31}$$

Using (2.29), (2.30) and (2.31) with $u = e^{-tA_{\alpha,\beta}}\varphi$, $t > 0$, we get that there exists a constant $C > 0$ such that

$$\begin{aligned}\|e^{-tA_{\alpha,\beta}}\varphi\|^2_{L^{2^\star}(\Omega\setminus\Sigma)} &\le C\mathcal{E}_{\alpha,\beta}(e^{-tA_{\alpha,\beta}}\varphi, e^{-tA_{\alpha,\beta}}\varphi)\\ &= C\left(A_{\alpha,\beta}e^{-tA_{\alpha,\beta}}\varphi, e^{-tA_{\alpha,\beta}}\varphi\right)_{L^2(\Omega\setminus\Sigma)}\\ &\le C\|A_{\alpha,\beta}e^{-tA_{\alpha,\beta}}\varphi\|_{L^2(\Omega\setminus\Sigma)}\|\varphi\|_{L^2(\Omega\setminus\Sigma)}\\ &\le \frac{C}{t}\|\varphi\|^2_{L^2(\Omega\setminus\Sigma)}\end{aligned}$$

for all $t > 0$ and $\varphi \in L^2(\Omega\setminus\Sigma)$. Therefore, $e^{-tA_{\alpha,\beta}}$ maps $L^2(\Omega\setminus\Sigma)$ into $L^{2^\star}(\Omega\setminus\Sigma)$ with

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^2(\Omega\setminus\Sigma),L^{2^\star}(\Omega\setminus\Sigma))} \le Ct^{-\frac{1}{2}}. \tag{2.32}$$

We claim that the estimate (2.32) extrapolates and gives the following estimate

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^1(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))} \le Ct^{-\frac{N}{2\beta}} \tag{2.33}$$

uniformly for all $t > 0$, for some constant $C > 0$. We proceed as in the proof of [43, Lemma 6.1]. By the Riesz-Thorin interpolation theorem [7, Section 1.1.5], we get from (2.32) that for every $p \in [2,\infty)$ and $t > 0$,

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}\left(L^p(\Omega\setminus\Sigma),L^{\frac{2^\star p}{2}}(\Omega\setminus\Sigma)\right)} \le C^{\frac{2}{p}}t^{-\frac{1}{p}}. \tag{2.34}$$

Let $t_k := \frac{2^\star-1}{2^\star}(2^\star)^{-k}$ and $p_k := 2\left(\frac{2^\star}{2}\right)^k$ for $k \ge 0$. Then

$$\sum_{k=0}^\infty t_k = 1 \quad \text{and} \quad \sum_{k=0}^\infty \frac{1}{p_k} = \frac{2^\star}{2(2^\star-2)} = \frac{N}{4\beta}.$$

Applying the estimate (2.34) with $p = p_k$ yields

$$\begin{aligned}\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^2(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))} &\le \prod_{k=0}^\infty \|e^{-t_kA_{\alpha,\beta}}\|_{\mathcal{L}(L^{p_k}(\Omega\setminus\Sigma),L^{p_{k+1}}(\Omega\setminus\Sigma))}\\ &\le \prod_{k=0}^\infty C^{\frac{2}{p_k}}t^{-\frac{1}{p_k}}t_k^{-\frac{1}{p_k}} = Ct^{-\frac{N}{4\beta}}.\end{aligned} \tag{2.35}$$

By duality, we deduce from (2.35) that

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^1(\Omega\setminus\Sigma),L^2(\Omega\setminus\Sigma))} \le Ct^{-\frac{N}{4\beta}}. \tag{2.36}$$

Combining (2.35) and (2.36) we get (2.33) and the claim is proved. Next, proceeding as in [43, Lemma 6.5] by using the estimate

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^2(\Omega\setminus\Sigma))} \leq Ce^{-\lambda_1 t}, \quad \text{for some } C > 0, \ \forall\, t \geq 0,$$

we observe that (2.33) improves to

$$\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^1(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))} \leq Ct^{-\frac{N}{2\beta}}e^{-\lambda_1 t}.$$

Next, let $\frac{1}{q} = \frac{\tau}{1} + \frac{1-\tau}{\infty}$, i.e., $\tau = \frac{1}{q}$. By the Riesz–Thorin interpolation theorem we infer

$$\begin{aligned}\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^q(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))} &\leq \|e^{-tA_{\alpha,\beta}}\|^{\tau}_{\mathcal{L}(L^1(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))}\|e^{-tA_{\alpha,\beta}}\|^{1-\tau}_{\mathcal{L}(L^\infty(\Omega\setminus\Sigma))}\\ &\leq C^{\frac{1}{q}}t^{-\frac{N}{2\beta}\frac{1}{q}}e^{-\frac{\lambda_1}{q}t}. \end{aligned} \tag{2.37}$$

Finally, let $\frac{1}{p} = \frac{\eta}{q} + \frac{1-\eta}{\infty}$, i.e., $\eta = \frac{q}{p}$. It follows from the Riesz–Thorin interpolation theorem and (2.37) that

$$\begin{aligned}\|e^{-tA_{\alpha,\beta}}\|_{\mathcal{L}(L^q(\Omega\setminus\Sigma),L^p(\Omega\setminus\Sigma))} &\leq \|e^{-tA_{\alpha,\beta}}\|^{\eta}_{\mathcal{L}(L^q(\Omega\setminus\Sigma))}\|e^{-tA_{\alpha,\beta}}\|^{1-\eta}_{\mathcal{L}(L^q(\Omega\setminus\Sigma),L^\infty(\Omega\setminus\Sigma))}\\ &\leq \left(C^{\frac{1}{q}}t^{-\frac{N}{2\beta}\frac{1}{q}}e^{-\frac{\lambda_1}{q}t}(1+t)^{\frac{N}{2\beta}\frac{1}{q}}\right)^{1-\frac{q}{p}}\\ &= C^{\frac{1}{q}-\frac{1}{p}}t^{-\frac{N}{2\beta}(\frac{1}{q}-\frac{1}{p})}e^{-\lambda_1(\frac{1}{q}-\frac{1}{p})t},\end{aligned}$$

and we have shown (2.27).

(b) If $\mathfrak{b} = 0$, then there exists a constant $C > 0$ such that for every $u \in D(\mathcal{E}_{\alpha,\beta})$,

$$\|u\|^2_{L^{2^\star}(\Omega\setminus\Sigma)} \leq C\left(\mathcal{E}_{\alpha,\beta}(u,u) + \|u\|^2_{L^2(\Omega\setminus\Sigma)}\right).$$

The proof of (2.28) follows the lines of the proof of part (a).

Next, recall that the semigroup $(e^{-tA_{\alpha,\beta}})_{t\geq 0}$ is compact on $L^2(\Omega\setminus\Sigma)$. Since Ω is bounded, the latter property together with the ultracontractivity estimates imply that the semigroup on $L^p(\Omega\setminus\Sigma)$ is compact for every $p \in [1,\infty]$ (see, e.g [7, Theorem 1.6.4]). Finally let u_k be an eigenfunction associated with λ_k. Then by definition, $u_k \in D\left(A_{\alpha,\beta}\right)$. Since the semigroup $(e^{-tA_{\alpha,\beta}})_{t\geq 0}$ is ultracontractive and $|\Omega\setminus\Sigma| = |\Omega| < \infty$, it follows from [7, Theorem 2.1.4] that $u_k \in L^\infty(\Omega\setminus\Sigma)$. This completes the proof of the theorem. □

Clearly, $S_{\alpha,\beta}(t) := e^{-tA_{\alpha,\beta}}$ defines a bounded (linear) operator from $L^q(\Omega\setminus\Sigma)$ into $L^p(\Omega\setminus\Sigma)$ for $1 \leq q \leq p \leq \infty$. For the sake of brevity, in what follows we may write (and define) its operator norm

$$\|S_{\alpha,\beta}(t)\|_{p,q} := \sup_{\|f\|_{L^q(\Omega\setminus\Sigma)}\leq 1}\left(\|S_{\alpha,\beta}(t)f\|_{L^p(\Omega\setminus\Sigma)}\right).$$

Of course, we have

$$\left\|S_{\alpha,\beta}(t)f\right\|_{L^p(\Omega\setminus\Sigma)} \leq \|S_{\alpha,\beta}(t)\|_{p,q}\|f\|_{L^q(\Omega\setminus\Sigma)},$$

for all $t > 0$ and $f \in L^q(\Omega\setminus\Sigma)$, and

$$\|S_{\alpha,\beta}(t)\|_{p,q} \leq C(t\wedge 1)^{-\frac{N}{2\beta}\left(\frac{1}{q}-\frac{1}{p}\right)}, \tag{2.38}$$

in all cases $\mathfrak{b} \geq 0$ (see Theorem 2.16).

Next we give an important embedding result.

Lemma 2.17. *Let $p \in (1, \infty)$. Then $D(A^{\theta}_{\alpha,\beta,p})$ embeds continuously into $L^{\infty}(\Omega \backslash \Sigma)$ provided that $\theta \in (0, 1]$ is a real number such that $\theta > N/(2\beta p)$.*

Proof. The proof is analogous to the proof of [15, Theorem 2.5, (d)] and follows in light of the ultracontractivity estimates for $S_{\alpha,\beta}(t)$, when it acts as a bounded operator from $L^p(\Omega \backslash \Sigma)$ into $L^{\infty}(\Omega \backslash \Sigma)$. □

We conclude this (sub)section by giving a more precise characterization of $A_{\alpha,\beta,p}$ at least for $p \geq 2$.

Proposition 2.18. *Let $p \in [2, \infty)$. Then*

$$D(A_{\alpha,\beta,p}) = \Big\{u \in D(\mathcal{E}_{\alpha,\beta}) \cap L^p(\Omega \backslash \Sigma),\ (-\Delta)^{\alpha}_{\Omega_-} u_- \in L^p(\Omega_-),\ (-\Delta)^{\beta}_{\Omega_+} u_+ \in L^p(\Omega_+),$$
$$\mathcal{N}^{2-2\alpha} u_- = 0 \ \text{ on } \ \Gamma,\ \mathcal{N}_w(u_+, u_-) + \mathfrak{b} u_- = 0 \ \text{ on } \ \Sigma\Big\} \tag{2.39}$$

and, for $u \in D\left(A_{\alpha,\beta,p}\right)$,

$$A_{\alpha,\beta,p} u = (-\Delta)^{\beta}_{\Omega_+} u_+ \ \text{on } \Omega_+, \ \text{and } A_{\alpha,\beta,p} u = (-\Delta)^{\alpha}_{\Omega_-} u_- \ \text{on } \Omega_-. \tag{2.40}$$

Proof. Let $p \geq 2$. Recall that $L^p(\Omega \backslash \Sigma) \hookrightarrow L^q(\Omega \backslash \Sigma)$ for all $1 \leq q \leq p \leq \infty$. It follows from the untracontractivity of the semigroup on $L^2(\Omega \backslash \Sigma)$ (Theorem 2.16) that $L^p(\Omega \backslash \Sigma)$ is invariant under the operator $e^{-tA_{\alpha,\beta}}$ for every $t > 0$. Thus $A_{\alpha,\beta,p}$ is the part of $A_{\alpha,\beta}$ in $L^p(\Omega \backslash \Sigma)$. Hence $A_{\alpha,\beta,p}$ is given by (2.39) and (2.40). □

Remark 2.19. We mention that if $1 \leq p < 2$, then a characterization of $A_{\alpha,\beta,p}$ as the one given in (2.39) and (2.40) is not an easy task even if one assumes that $\Omega \backslash \Sigma$ is a smooth open set. This is in part due to the fact that fine elliptic regularity for the regional fractional Laplace operator with fractional Neumann boundary conditions is not yet available in the literature, and is generally difficult to establish. But a partial characterization of $A_{\alpha,\beta,p}$ for $1 \leq p < 2$ can be obtained from the general case of operators generated by Dirichlet forms contained in [43, Theorem 3.9].

2.4. The linear elliptic problem

Let Ω, Ω_+, Ω_-, Σ be as in Figure 1 and $1/2 < \beta \leq \alpha < 1$. In this (sub)section, we consider the following elliptic boundary value problem

$$\begin{cases} (-\Delta)^{\beta}_{\Omega_+} u_+ = f_+ & \text{in } \ \Omega_+, \\ (-\Delta)^{\alpha}_{\Omega_-} u_- = f_- & \text{in } \ \Omega_-, \\ \mathcal{N}^{2-2\alpha} u_- = 0 & \text{on } \ \Gamma, \\ u_+ = w u_- & \text{on } \ \Sigma, \\ \mathcal{N}_w(u_+, u_-) + \mathfrak{b} u_- = 0 & \text{on } \ \Sigma. \end{cases} \tag{2.41}$$

Here, we only assume $f_\pm = f_\pm(x)$ and let w, $\mathfrak{b}$ be the same as in the previous (sub)sections.

Definition 2.20. A function $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$ is said to be a *weak solution of* (2.41) if the equality

$$\mathcal{E}_{\alpha,\beta}(u,v) = \langle f_+, v_+\rangle_{\beta,+} + \langle f_-, v_-\rangle_{\alpha,-}, \tag{2.42}$$

holds for every $v \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$, where $\langle\cdot,\cdot\rangle_{\beta,+}$ (resp. $\langle\cdot,\cdot\rangle_{\alpha,-}$) denotes the duality between $W^{\beta,2}(\Omega_+)$ and $(W^{\beta,2}(\Omega_+))^\star$ (resp. $W^{\alpha,2}(\Omega_-)$ and $(W^{\alpha,2}(\Omega_-))^\star$).

We have the following result of existence of weak solutions.

Proposition 2.21. *The following assertions hold.*

(a) *Let* $\mathfrak{b}$ *satisfy* (2.22). *Then for every* $f_+ \in (W^{\beta,2}(\Omega_+))^\star$) *and* $f_- \in (W^{\alpha,2}(\Omega_-))^\star$, *there exists a unique weak solution* u *of* (2.41). *Moreover, there exists a constant* $C > 0$ *such that*

$$\|u\|_{\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)} \le C\left(\|f_+\|_{(W^{\beta,2}(\Omega_+))^\star} + \|f_-\|_{(W^{\alpha,2}(\Omega_-))^\star}\right). \tag{2.43}$$

(b) *Let* $\mathfrak{b} = 0$ *a.e. on* Σ. *Then for every* $f_+ \in (W^{\beta,2}(\Omega_+))^\star$) *and* $f_- \in (W^{\alpha,2}(\Omega_-))^\star$ *satisfying*

$$\langle f_+, 1\rangle_{\beta,+} + \langle f_-, 1\rangle_{\alpha,-} = 0, \tag{2.44}$$

there exists a weak solution u *of* (2.41). *Moreover, there exists a constant* $C > 0$ *such that*

$$\|u - \overline{u}_+\|_{W^{(\alpha,\beta),2}(\Omega\backslash\Sigma)} \le C\left(\|f_+\|_{(W^{\beta,2}(\Omega_+))^\star} + \|f_-\|_{(W^{\alpha,2}(\Omega_-))^\star}\right), \tag{2.45}$$

where $\overline{u}_+ := \frac{1}{\sigma(\Sigma)}\int_\Sigma u_+ d\sigma$.

Proof.

(a) Assume that $\mathfrak{b}$ satisfies (2.22). It follows from the embeddings (2.10) and (2.11) that $(W^{\alpha,2}(\Omega_-))^\star \hookrightarrow (\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma))^\star$ and $(W^{\beta,2}(\Omega_+))^\star \hookrightarrow (\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma))^\star$. Thus it is clear that the right hand side of (2.42) defines a continuous linear functional on $\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$. Moreover we have shown that the bilinear form $\mathcal{E}_{\alpha,\beta}$ is continuous and elliptic. Then $\mathcal{E}_{\alpha,\beta}$ is continuous, elliptic and coercive (by using (2.12)). Hence, by the Lax–Milgram lemma, for every $f_+ \in (W^{\beta,2}(\Omega_+))^\star$) and $f_- \in (W^{\alpha,2}(\Omega_-))^\star$, the system (2.41) has a unique weak solution $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$. Taking $v = u$ as a test function in (2.42), the estimate (2.43) follows from the fact that there exists a constant $C > 0$ such that

$$C\|u\|^2_{\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)} \le \mathcal{E}_{\alpha,\beta}(u,u) \le \left(\|f_+\|_{(W^{\beta,2}(\Omega_+))^\star} + \|f_-\|_{(W^{\alpha,2}(\Omega_-))^\star}\right)\|u\|_{\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)}.$$

(b) Assume $\mathfrak{b} = 0$ a.e. on Σ. Then taking $v = 1$ as a test function in (2.42) we have that (2.44) is a necessary condition for the existence of weak solutions. Let

$$\mathbb{W}^{(\alpha,\beta),2,0}(\Omega\backslash\Sigma) := \{u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) : \overline{u}_+ = 0\},$$

and define the bilinear form $\mathcal{E}_{\alpha,\beta,0}$ with domain $D(\mathcal{E}_{\alpha,\beta,0}) := \mathbb{W}^{(\alpha,\beta),2,0}(\Omega\backslash\Sigma)$ and given by

$$\mathcal{E}_{\alpha,\beta,0}(u,v) = \mathcal{E}_{\alpha,\beta}(u,v). \tag{2.46}$$

Then $\mathcal{E}_{\alpha,\beta,0}$ is a closed, continuous and coercive form and the right hand side of (2.42) defines a continuous functional on $\mathbb{W}^{(\alpha,\beta),2,0}(\Omega\backslash\Sigma)$. Hence for every $f_+ \in (W^{\beta,2}(\Omega_+))^\star$) and $f_- \in (W^{\alpha,2}(\Omega_-))^\star$ satisfying (2.44), the system (2.41) has a unique weak solution

$u_0 \in W^{(\alpha,\beta),2,0}(\Omega\setminus\Sigma)$. Let $u_0 = u - \overline{u}_+ \in \mathbb{W}^{(\alpha,\beta),2,0}(\Omega\setminus\Sigma)$ for some $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\setminus\Sigma)$, and notice that $\mathcal{E}_{\alpha,\beta}(u-\overline{u}_+, v) = \mathcal{E}_{\alpha,\beta}(u, v)$ for every $v \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\setminus\Sigma)$. Moreover, there is a constant $C > 0$ such that

$$\|u - \overline{u}_+\|^2_{\mathbb{W}^{(\alpha,\beta),2}(\Omega\setminus\Sigma)} \le C\mathcal{E}_{\alpha,\beta}(u - \overline{u}_+, u - \overline{u}_+).$$

Thus taking $v = u - \overline{u}_+$ as a test function in (2.42) we get the estimate (2.45) and the proof is finished. □

The following theorem is the main result of this (sub)section. The first two statements yield optimal conditions for the existence of bounded solutions while the last one is an interior Hölder continuity result.

Theorem 2.22. *Let $f_+ \in L^q(\Omega_+)$ and $f_- \in L^q(\Omega_-)$ for some $q > 1$ and let u be a weak solution of the system (2.41). Then the following assertions hold.*

(a) *If $q > \frac{N}{2\beta}$ and $\mathfrak{b}$ satisfies (2.22), then $u \in L^\infty(\Omega\setminus\Sigma)$ and there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\Omega\setminus\Sigma)} \le C\left(\|f_+\|_{L^q(\Omega_+)} + \|f_-\|_{L^q(\Omega_-)}\right). \tag{2.47}$$

(b) *If $q > \frac{N}{2\beta}$, $\mathfrak{b} = 0$ a.e. on Σ and*

$$\int_{\Omega_+} f_+ dx + \int_{\Omega_-} f_- dx = 0, \tag{2.48}$$

then $u \in L^\infty(\Omega\setminus\Sigma)$ and there exists a constant $C > 0$ such that

$$\|u - \overline{u}_+\|_{L^\infty(\Omega\setminus\Sigma)} \le C\left(\|f_+\|_{L^q(\Omega_+)} + \|f_-\|_{L^q(\Omega_-)}\right). \tag{2.49}$$

(c) *If $q = \infty$ and $\mathfrak{b}$ satisfies (2.22), or $\mathfrak{b} = 0$ and f_+, f_- satisfy (2.48), then for any sufficiently small $\xi > 0$, there exists a constant $C > 0$ depending only N, α, β and ξ such that*

$$\|u_+\|_{C^{0,2\beta}((\overline{\Omega_+})_\xi)} + \|u_-\|_{C^{0,2\alpha}((\overline{\Omega_-})_\xi)} \le C\left(\|f\|_{L^\infty(\Omega\setminus\Sigma)} + \|u\|_{L^\infty(\Omega\setminus\Sigma)}\right), \tag{2.50}$$

where $(\Omega_+)_\xi := \{x \in \Omega_+ : \ dist(x, \partial\Omega_+) > \xi\}$ and $(\Omega_-)_\xi := \{x \in \Omega_- : \ dist(x, \partial\Omega_-) > \xi\}$.

Proof.
(a) Assume $q > \frac{N}{2\beta}$ and that $\mathfrak{b}$ satisfies (2.22). Then by Proposition 2.12 the operator $A_{\alpha,\beta}$ is invertible and the solution u is given by $u = A^{-1}_{\alpha,\beta} f$. Using the ultracontractivity estimate (2.27) with $p = \infty$ and the fact that the operator resolvent $A^{-1}_{\alpha,\beta}$ is the Laplace transform of the semigroup, we get that there exists a constant $C > 0$ such that for a.e. $x \in \Omega\setminus\Sigma$,

$$\begin{aligned} |u(x)| = |A^{-1}_{\alpha,\beta} f(x)| &\le \int_0^\infty |e^{-tA_{\alpha,\beta}} f(x)| \, dt \\ &\le C \int_0^\infty e^{-\frac{\lambda_1}{q}t} t^{-\frac{N}{2\beta q}} \, dt \|f\|_{L^q(\Omega\setminus\Sigma)} \\ &\le C\left(\int_1^\infty e^{-\frac{\lambda_1}{q}t} t^{-\frac{N}{2\beta q}} \, dt + \int_0^1 e^{-\frac{\lambda_1}{q}t} t^{-\frac{N}{2\beta q}} \, dt\right) \|f\|_{L^q(\Omega\setminus\Sigma)}. \end{aligned} \tag{2.51}$$

The first integral in the right hand side of (2.51) is always finite. Since $q > \frac{N}{2\beta}$, we have that the second integral is also finite. Hence, (2.47) follows from (2.51).

(b) Assume $q > \frac{N}{2\beta}$ and that $\mathfrak{b} = 0$. Proceeding as in part (a) and the proof of Proposition 2.21(b) we get the estimate (2.49).

(c) Now assume that $q = \infty$. Then by part (a) if $\mathfrak{b}$ satisfies (2.22), and by part (b) if $\mathfrak{b} = 0$, the system (2.41) has a weak solution $u \in \mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \cap L^\infty(\Omega\backslash\Sigma)$. Define the functions

$$\widetilde{u}_+ = \begin{cases} u_+ & \text{in } \Omega_+ \\ 0 & \text{in } \mathbb{R}^N\backslash\Omega_+ \end{cases} \quad \text{and} \quad \widetilde{u}_- = \begin{cases} u_- & \text{in } \Omega_- \\ 0 & \text{in } \mathbb{R}^N\backslash\Omega_-. \end{cases}$$

Note that $\widetilde{u}_+ \in L^\infty(\mathbb{R}^N) \hookrightarrow \mathcal{L}^1_\beta(\mathbb{R}^N)$ and $\widetilde{u}_- \in L^\infty(\mathbb{R}^N) \hookrightarrow \mathcal{L}^1_\alpha(\mathbb{R}^N)$. Moreover, using (2.3) we have that

$$(-\Delta)^\beta \widetilde{u}_+ = f_+ + V_{\Omega_+}(x) u_+ \ \text{ and } \ (-\Delta)^\alpha \widetilde{u}_- = f_- + V_{\Omega_-}(x) u_-,$$

where $V_{\Omega_+}(x)$ and $V_{\Omega_-}(x)$ denotes the potential given by (2.2). It follows from [41, Lemma 3.2] that for $\xi > 0$ sufficiently small, there exists a constant $C > 0$ depending on N, β and ξ such that

$$\begin{aligned} \|u_+\|_{C^{0,2\beta}(\overline{(\Omega_+)_\xi})} &\leq C\|\widetilde{u}_+\|_{L^\infty(\mathbb{R}^N)} + \|f_+ + V_{\Omega_+} u_+\|_{L^\infty((\Omega_+)_{\frac{\xi}{2}})} \\ &\leq C\left(\|u_+\|_{L^\infty(\Omega_+)} + \|f_+\|_{L^\infty(\Omega_+)}\right). \end{aligned} \tag{2.52}$$

Similarly, we obtain

$$\|u_-\|_{C^{0,2\alpha}(\overline{(\Omega_-)_\xi})} \leq C\left(\|u_-\|_{L^\infty(\Omega_-)} + \|f_-\|_{L^\infty(\Omega_-)}\right). \tag{2.53}$$

Combining (2.52) and (2.53) we arrive at estimate (2.50) and the proof is finished. □

3. Well-posedness of mild solutions

In the present section, we rely on the crucial results of Section 2 (in particular, those of Theorem 2.16) to develop well-posedness results in the same spirit of [44] where second order elliptic operators in divergence form have been considered. To this end, we need to introduce some further notations and basic definitions. Let $T > 0$ be fixed but otherwise arbitrary, $p \in [1, \infty]$ and $\delta \in [0, \infty)$. We begin with defining the Banach space

$$E_{p,\delta,T} := \Big\{ u : (\Omega\backslash\Sigma) \times (0, T] \to \mathbb{R} \ \text{ measurable, } u(\cdot, t) \in L^p(\Omega\backslash\Sigma) \ \text{ for a.e } t \in (0, T],$$
$$\|u\|_{E_{p,\delta,T}} = |||u|||_{p,\delta,T} := \sup_{t\in(0,T]} (t \wedge 1)^\delta \|u(\cdot, t)\|_{L^p(\Omega\backslash\Sigma)} < \infty \Big\}.$$

As in the previous section, by $u \in L^p(\Omega\backslash\Sigma)$, we understand that $u|_{\Omega_+} = u_+ \in L^p(\Omega_+)$ and $u|_{\Omega_-} = u_- \in L^p(\Omega_-)$. The same principle is applied to other functions defined over $\Omega\backslash\Sigma$. We also introduce the Banach space

$$L_{p_1,p_2,T} := \Big\{ u : (\Omega\backslash\Sigma) \times (0, T] \to \mathbb{R} \ \text{ measurable such that}$$
$$\|u\|_{L_{p_1,p_2,T}} = \|u\|_{p_1,p_2,T} := \sup_{t_1,t_2\in[0,T], 0\leq t_2-t_1\leq 1} \left(\int_{t_1}^{t_2} \|u\|^{p_2}_{L^{p_1}(\Omega\backslash\Sigma)} \, d\tau \right)^{\frac{1}{p_2}} < \infty \Big\},$$

for $p_1, p_2 \in [1, \infty)$, with the obvious modifications when $p_1 = p_2 = \infty$.

We can cast the semilinear parabolic system (1.8)–(1.12) into functional form by defining the function

$$F(u) := \begin{cases} -f_+(u_+) & \text{on } \Omega_+ \\ -f_-(u_-) & \text{on } \Omega_-. \end{cases}$$

In this case, for $u \in D\left(A_{\alpha,\beta}\right)$ we can conveniently rewrite the system (1.8)–(1.12) as follows:

$$\partial_t u + A_{\alpha,\beta} u = F(u) \text{ in } (\Omega\backslash\Sigma) \times (0,T],\ u(\cdot,0) = u_0 \text{ in } \Omega\backslash\Sigma. \tag{3.1}$$

Our main goal in this section is to state sufficiently general conditions on F (and therefore on the nonlinear functions $f_\pm$) for which we can infer the existence of properly-defined solutions for (3.1). Once again, let $T \in (0,\infty)$ and denote by I a time interval of the form $[0,T]$, $[0,T)$ or $[0,\infty)$.

Definition 3.1. By a *mild solution of* (3.1) *on the interval* I, we mean that the measurable function u has the following properties:

(a) $u(\cdot,t) \in L^1(\Omega\backslash\Sigma)$, for all $t \in I\backslash\{0\}$.

(b) $F(u(\cdot,t)) \in L^1(\Omega\backslash\Sigma)$, for almost all $t \in I\backslash\{0\}$.

(c) $\int_0^t \|F(u(\cdot,s))\|_{L^1(\Omega\backslash\Sigma)}\, ds < \infty$, for all $t \in I$.

(d) $u(\cdot,t) = S_{\alpha,\beta}(t)\, u_0 + \int_0^t S_{\alpha,\beta}(t-s)\, F(u(\cdot,s))\, ds$, for all $t \in I\backslash\{0\}$, where the integral is an absolutely converging Bochner integral in the space $L^1(\Omega\backslash\Sigma)$.

(e) The initial datum u_0 is assumed in the following sense:

$$\lim_{t\to 0^+} \|u(\cdot,t) - u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} = 0,$$

for some $u_0 \in L^{p_0}(\Omega\backslash\Sigma)$, if $p_0 \in [1,\infty)$, and $u_0 \in X^\infty(\Omega\backslash\Sigma) := \overline{D(A_{\alpha,\beta,\infty})}^{L^\infty(\Omega\backslash\Sigma)}$ if $p_0 = \infty$.

Remark 3.2. By Proposition 2.14 and Remark 2.15, the semigroup $S_{\alpha,\beta}$ is strongly continuous on $L^{p_0}(\Omega\backslash\Sigma)$, if $p_0 \in [1,\infty)$. We recall that the semigroup is not strongly continuous on $L^\infty(\Omega\backslash\Sigma)$, but by definition we also have that $S_{\alpha,\beta}$ is strongly continuous on $X^\infty(\Omega\backslash\Sigma)$. For simplicity of notation, throughout the following, for $p_0 \in [1,\infty]$, we will sometimes also denote

$$X^{p_0}(\Omega\backslash\Sigma) = L^{p_0}(\Omega\backslash\Sigma) \text{ if } p_0 \in [1,\infty).$$

Thus for every $p_0 \in [1,\infty]$ and $u_0 \in X^{p_0}(\Omega\backslash\Sigma)$, we have that

$$\lim_{t\to 0^+} \left\|S_{\alpha,\beta}(t)\, u_0 - u_0\right\|_{L^{p_0}(\Omega\backslash\Sigma)} = 0.$$

We then observe that condition (e) of Definition 3.1 holds if and only if

$$\lim_{t\to 0^+} \left\|S_{\alpha,\beta}(t)\, u_0 - u(\cdot,t)\right\|_{L^{p_0}(\Omega\backslash\Sigma)} = 0.$$

Using Theorem 2.22 it is easy to see that $X^\infty(\Omega\backslash\Sigma)$ also coincides with $\overline{D(A_{\alpha,\beta,p})}^{L^\infty(\Omega\backslash\Sigma)}$, for $p > N/(2\beta)$. We conjecture that

$$X^\infty(\Omega\backslash\Sigma) = C_w(\overline{\Omega}) := \{u:\ u_+ \in C(\overline{\Omega}_+),\ u_- \in C(\overline{\Omega}_-) \text{ and } u_+ = wu_- \text{ on } \Sigma\}. \tag{3.2}$$

In fact, the important step to get (3.2) is to show that $S_{\alpha,\beta}(t)$ leaves $C_w(\overline{\Omega})$ invariant for every $t \geq 0$, or equivalently, that $D(A_{\alpha,\beta,\infty}) \subset C_w(\overline{\Omega})$. Since this type of regularity is not yet available in the literature, and is also not the main concern of the present paper, we will not further go into details.

The first theorem establishes the existence of locally defined mild solutions under some suitable assumptions on the nonlinear $f_\pm$. Let $\gamma \in [1,\infty)$ and $C_{f_\pm} > 0$. These assumptions are as follows.

(F1) $f_\pm : \mathbb{R} \to \mathbb{R}$ is measurable function such that

$$\left|f_\pm(\xi)\right| \leq C_{f_\pm}\left(1 + |\xi|^\gamma\right), \text{ for all } \xi \in \mathbb{R}.$$

(F2) For all $\xi, \eta \in \mathbb{R}$, assume the local Lipschitz condition

$$\left|f_\pm(\xi) - f_\pm(\eta)\right| \leq C_{f_\pm}\left(1 + |\xi| + |\eta|\right)^{\gamma-1}|\xi - \eta|.$$

(F3) There exists a positive increasing function $Q_\pm : \mathbb{R} \to \mathbb{R}_+$ such that

$$\left|f_\pm(\xi)\right| \leq Q_\pm(|\xi|), \quad \text{for all } \xi \in \mathbb{R}.$$

(F4) For all $\xi, \eta \in \mathbb{R}$, assume the local Lipschitz condition

$$\left|f_\pm(\xi) - f_\pm(\eta)\right| \leq Q_\pm(|\xi| + |\eta|)\,|\xi - \eta|.$$

We notice that conditions (**F3**) and (**F4**) are more general alternatives to (**F1**) and (**F2**), respectively, and shall be imposed only in some situations.

We employ a contraction argument in the Banach space $E_{p,\delta,T}$, $p \in [1,\infty]$, $p \geq p_0$, (with a singularity at $t = 0$) to construct a solution u locally in time. In what follows, let

$$\mathfrak{n} := \frac{N}{2\beta} \text{ for } N \geq 2, \quad \frac{1}{2} < \beta < 1 \text{ and } \delta := \frac{\mathfrak{n}}{p_0} - \frac{\mathfrak{n}}{p} \in [0,\infty).$$

Theorem 3.3 (Local existence). *Assume either one of the following.*

(a) *Assume* **(F1)** *and* **(F2)** *for some* $\gamma \in [1,\infty)$ *and let* $u_0 \in L^{p_0}(\Omega\backslash\Sigma)$, *for some* $p_0 \in [1,\infty)$ *such that*

$$(\gamma - 1)\frac{\mathfrak{n}}{p_0} < 1.$$

(b) *Assume* **(F3)** *and* **(F4)** *and let* $u_0 \in X^\infty(\Omega\backslash\Sigma)$.

(c) *Assume* **(F1)** *and* **(F2)** *for some* $\gamma \in (1,\infty)$ *and let* $u_0 \in L^{p_0}(\Omega\backslash\Sigma)$, *for some* $p_0 \in (1,\infty)$ *such that*

$$(\gamma - 1)\frac{\mathfrak{n}}{p_0} = 1.$$

Then there exists a time $T > 0$ (depending on u_0) such that the initial value problem (3.1) *has a unique mild solution in the sense of Definition 3.1 on the interval* $[0, T]$.

The assertion of Theorem 3.3 with assumptions (a) and (c) follows from the validity of the next two lemmas. Lemma 3.4 deals with the case when $(\gamma - 1)\,\mathfrak{n}/p_0 < 1$ while the second one considers the limiting case $(\gamma - 1)\,\mathfrak{n}/p_0 = 1$ (see Lemma 3.5). The second case (b) is treated in Lemma 3.6.

Lemma 3.4. *Let $p_0 \in [1,\infty)$ and assume that the hypothesis (a) of Theorem 3.3 is satisfied for some $\gamma \in [1,\infty)$. Then the assertion of Theorem 3.3 holds.*

Proof. The proof is inspired from the proof of [44, Lemma 7] and is developed using the crucial ultracontractivity estimates of Theorem 2.16 (see also (2.38)). In this proof and elsewhere, the constant $C > 0$ is independent of the times t, s, T. We shall explicitly state its further dependence on other parameters whenever necessary. We also preliminarily observe that by **(F1)** and **(F2)** we have

$$|F(\xi)| \leq C_F \left(1 + |\xi|^\gamma\right), \ \forall \xi \in \mathbb{R}, \tag{3.3}$$

as well as the following,

$$|F(\xi) - F(\eta)| \leq C_F (1 + |\xi| + |\eta|)^{\gamma-1} |\xi - \eta|, \ \forall \xi, \eta \in \mathbb{R}. \tag{3.4}$$

Here, $C_F = \max\left\{C_{f_+}, C_{f_-}\right\}$.

Let now $p \in [1,\infty)$ such that $p \geq p_0$ and

$$\gamma \leq p, \quad \gamma\delta < 1 \text{ and } (\gamma - 1)\left(\delta + \frac{\mathfrak{n}}{p}\right) + \varepsilon < 1, \tag{3.5}$$

for some $\varepsilon \in (0,1)$. The proof exploits a Picard iteration argument. To this end, let $T \in (0,\infty)$ and fix an element $u_1 \in E_{p,\delta,T}$ which is otherwise arbitrary. We define a sequence

$$u_{m+1}(\cdot,t) = S_{\alpha,\beta}(t)\, u_0 + \int_0^t S_{\alpha,\beta}(t-s)\, F(u_m(\cdot,s))\, ds, \ t \in (0,T], \tag{3.6}$$

for all $m \in \mathbb{N}$. We first show by induction that $u_m \in E_{p,\delta,T}$, for all $m \in \mathbb{N}$. To this end, let $s_1 \in [1,\infty)$ be such that

$$\frac{\gamma}{p} \leq \frac{1}{s_1} \quad \text{and} \quad \frac{\mathfrak{n}}{s_1} + (\gamma - 1)\,\delta + \varepsilon < 1 + \frac{\mathfrak{n}}{p},$$

and suppose that $u_m \in E_{p,\delta,T}$ is already known. The bound (3.3) and the Hölder inequality yield

$$\begin{aligned}
(t \wedge 1)^\delta \|u_{m+1}(\cdot,t)\|_{L^p(\Omega\setminus\Sigma)} &\leq (t \wedge 1)^\delta \left\|S_{\alpha,\beta}(t)\, u_0\right\|_{L^p(\Omega\setminus\Sigma)} \\
&\quad + (t \wedge 1)^\delta \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1} \|F(u_m(\cdot,s))\|_{L^{s_1}(\Omega\setminus\Sigma)}\, ds \\
&\leq (t \wedge 1)^\delta \left\|S_{\alpha,\beta}(t)\right\|_{p,p_0} \|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} \\
&\quad + (t \wedge 1)^\delta \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1} C_F (s \wedge 1)^{-\gamma\delta} \\
&\quad \times \left[(s \wedge 1)^\delta \left(\|1 + |u_m(\cdot,s)|\|_{L^p(\Omega\setminus\Sigma)}\right)\right]^\gamma ds.
\end{aligned} \tag{3.7}$$

The first term on the right-hand side of (3.7) can be estimated owing to (2.38) for $p \geq p_0$ and the definition of $\delta = \mathfrak{n}/p_0 - \mathfrak{n}/p$, $\mathfrak{n} := N/(2\beta)$. For the second term we apply Lemma A (see Appendix) with $f(s) \equiv C_F$, $\theta := \gamma\delta$ and $s_2 = \infty$ (we also note that $p_\infty\left(f\right) = C_F$), whose assumptions are satisfied since $\gamma\delta < 1$, $\gamma\delta + \varepsilon \leq 1 + \delta$ and

$$\frac{\mathfrak{n}}{s_1} < 1 + \frac{\mathfrak{n}}{p}, \qquad \frac{\mathfrak{n}}{s_1} + \gamma\delta + \varepsilon < 1 + \frac{\mathfrak{n}}{p} + \delta.$$

In the space $E_{p,\delta,T}$, from (3.7) and using (2.38) we get

$$
\begin{aligned}
|||u_{m+1}|||_{p,\delta,T} &\leq C\,\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} \\
&\quad + \sup_{t\in(0,T]}\left[(t\wedge 1)^{\delta}\int_0^t \left\|S_{\alpha,\beta}\,(t-s)\right\|_{p,s_1}(s\wedge 1)^{-\gamma\delta}\,ds\right]|||1+|u_m||||^{\gamma}_{p,\delta,T} \\
&\leq C\left(\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} + (T\wedge 1)^{\varepsilon}\,|||1+|u_m||||^{\gamma}_{p,\delta,T}\right), \qquad (3.8)
\end{aligned}
$$

for some constant $C > 0$ which also depends on C_F. Henceforth, $u_{m+1} \in E_{p,\delta,T}$ and the claim is proved. Analogously, exploiting the Lipschitz condition (3.4), the Hölder inequality together with the application of Lemma A as above, we also find the uniform estimate

$$
|||u_{m+1}-u_m|||_{p,\delta,T} \leq C\,(T\wedge 1)^{\varepsilon}\,|||1+|u_m|+|u_{m-1}||||^{\gamma-1}_{p,\delta,T}\,|||u_m-u_{m-1}|||_{p,\delta,T}, \qquad (3.9)
$$

for all $m \geq 2$. Define $U := |||u_1|||_{p,\delta,T} + 2\,|||u_2-u_1|||_{p,\delta,T}$ and choose a small enough time $T_* \in (0,1]$ such that

$$
C\,(T_*\wedge 1)^{\varepsilon}\,(1+2U)^{\gamma-1} \leq \frac{1}{2}. \qquad (3.10)
$$

It follows from (3.9) that

$$
\begin{cases}
|||u_m|||_{p,\delta,T_*} \leq U, & \text{for all } m \geq 1,\\
|||u_{m+1}-u_m|||_{p,\delta,T_*} \leq \frac{1}{2}\,|||u_m-u_{m-1}|||_{p,\delta,T_*}, & \text{for all } m \geq 2.
\end{cases} \qquad (3.11)
$$

Thus, by iteration in (3.11), the sequence $\{u_m\}_{m\in\mathbb{N}}$ is Cauchy in the Banach space E_{p,δ,T_*}. Thus, it has a limit $u \in E_{p,\delta,T_*}$ such that

$$
\lim_{m\to\infty} |||u_m-u|||_{p,\delta,T_*} = 0. \qquad (3.12)
$$

It now remains to show that the limit u has all the required properties of Definition 3.1(a)–(e) on the time interval $[0,T_*]$. Property (a) is immediate since $u \in E_{p,\delta,T_*}$; (b) and (c) follow from the estimate

$$
\begin{aligned}
\|F(u)\|_{1,1,T_*} &\leq \int_0^{T_*}\|F(u(\cdot,s))\|_{L^{s_1}(\Omega\setminus\Sigma)}\,ds \\
&\leq C_F\int_0^{T_*}(s\wedge 1)^{-\gamma\delta}\left[(s\wedge 1)^{\delta}\,\|1+|u(\cdot,s)|\|_{L^p(\Omega\setminus\Sigma)}\right]^{\gamma} ds \\
&\leq \frac{C_F}{1-\gamma\delta}\,(T_*)^{1-\gamma\delta}\,|||1+|u||||^{\gamma}_{p,\delta,T_*}, \qquad (3.13)
\end{aligned}
$$

owing to the bound (3.3), the Hölder inequality, the fact that $0 < T_* \leq 1$ and $0 < \gamma\delta < 1$. Similar reasoning, using the Lipschitz bound (3.4), the contractivity properties of $S_{\alpha,\beta}(t)$ and the Hölder inequality once more, yields for all $t \in (0,T_*]$,

$$
\begin{aligned}
&\left\|\int_0^t S_{\alpha,\beta}\,(t-s)\left(F(u_m(\cdot,s)) - F(u(\cdot,s))\right)ds\right\|_{L^1(\Omega\setminus\Sigma)} \\
&\leq \int_0^t \|F(u_m(\cdot,s)) - F(u(\cdot,s))\|_{L^{s_1}(\Omega\setminus\Sigma)}\,ds \\
&\leq C_F\int_0^t (s\wedge 1)^{-\gamma\delta}\,ds\,|||1+|u_m|+|u||||^{\gamma-1}_{p,\delta,T_*}\,|||u_m-u|||_{p,\delta,T_*} \\
&\leq \frac{C_F}{1-\gamma\delta}\,(T_*)^{1-\gamma\delta}\,|||1+|u_m|+|u||||^{\gamma-1}_{p,\delta,T_*}\,|||u_m-u|||_{p,\delta,T_*} \qquad (3.14)
\end{aligned}
$$

which converges to zero as $m \to \infty$, by (3.12). Both (3.12) and (3.14) allow us to take the limit in $L^1(\Omega\backslash\Sigma)$-norm as $m \to \infty$ in the integral equation (3.6) in order to deduce the integral equation in Definition 3.1(d). For the last property (e), by Remark 3.2 it suffices to check that

$$\lim_{t\to 0^+} \left\|u(\cdot,t) - S_{\alpha,\beta}(t)u_0\right\|_{L^{p_0}(\Omega\backslash\Sigma)} = 0.$$

To this end, let $s_0 \in [1,\infty)$ be such that $\gamma/p \leq 1/s_0$ and $\mathfrak{n}/s_0 + \gamma\delta + \varepsilon < 1 + \mathfrak{n}/p_0$. The subsequent computation is similar to (3.8) but now we apply the statement of Lemma A (see Appendix) with the choices $p := p_0$, $s_1 := s_0$, $s_2 := \infty$, $\theta := \gamma\delta$, $\delta := 0$ and $\varepsilon := \varepsilon$ (note again that $f(s) \equiv C_F$). Indeed, the bound (3.3) and by virtue of Hölder's inequality, for all $t \in (0, T_*]$ we have

$$\begin{aligned}\left\|u(\cdot,t) - S_{\alpha,\beta}u_0\right\|_{L^{p_0}(\Omega\backslash\Sigma)} &\leq C_F\left(\int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p_0,s_0}(s\wedge 1)^{-\gamma\delta}\,ds\right)|||1+|u|||_{p,\delta,T}^{\gamma}\\ &\leq C(t\wedge 1)^{\varepsilon}\,|||1+|u|||_{p,\delta,T}^{\gamma},\end{aligned} \tag{3.15}$$

which implies the desired assertion (e) of Definition 3.1.

The uniqueness of the mild solution follows from a similar computation which resembles (3.9). Indeed, let $T \in (0, T_*]$ and let $u_1, u_2 \in E_{p,\delta,T}$ be any two mild solutions of (3.1) corresponding to the same initial datum u_0. As in (3.9), we get

$$|||u_1 - u_2|||_{p,\delta,T} \leq C(T\wedge 1)^{\varepsilon}\,|||1+|u_1|+|u_2|||_{p,\delta,T}^{\gamma-1}\,|||u_1-u_2|||_{p,\delta,T}. \tag{3.16}$$

for all $T \in (0, T_*]$. Hence, there exists a small time $\widehat{T} \in (0, T_*]$ such that $u_1(\cdot,t) \equiv u_2(\cdot,t)$ for $t \in [0,\widehat{T}]$ and uniqueness over the whole interval $[0, T_*]$ follows by a standard continuation argument. The proof is finished. □

Lemma 3.5. *Let $p_0 \in (1,\infty)$ and assume that the hypothesis (c) of Theorem 3.3 is satisfied for some $\gamma \in (1,\infty)$. Then the assertion of Theorem 3.3 holds.*

Proof. Choose a value $p \geq p_0$, $p \in (1,\infty]$ such that

$$\gamma \leq p,\ \gamma\delta < 1 \text{ and } (\gamma-1)\left(\delta + \frac{\mathfrak{n}}{p}\right) = 1.$$

We may apply the whole statement of Lemma B with the following choices $p := p_0$, $q := p$ and the set $\Pi := \{u_0\} \subset L^{p_0}(\Omega\backslash\Sigma)$. Consider the functions g, W constructed in Lemma B and recall that $(W(t))^{-\delta} = g(t)(t\wedge 1)^{-\delta}$. The proof is in the same spirit of [44, Lemma 8] where in the proof of previous Lemma 3.4, we perform the uniform estimates in a new (weighted) Banach space $E_{W,p,\delta,T} \subset E_{p,\delta,T}$ given by

$$E_{W,p,\delta,T} := \left\{u \in E_{p,\delta,T},\ \|u\|_{W,p,\delta,T} := \sup_{t\in(0,T]}\left((W(t))^{\delta}\,\|u(\cdot,t)\|_{L^p(\Omega\backslash\Sigma)}\right) < \infty\right\}.$$

As we mentioned already, the proof is based on the same iteration argument performed for the sequence (3.6) taking place now in the space $E_{W,p,\delta,T}$. First, to show that $u_m \in E_{W,p,\delta,T}$ is well-defined for all $m \in \mathbb{N}$, we again apply an induction argument. Suppose that $u_1 \in E_{W,p,\delta,T}$ is arbitrary and assume that $u_m \in E_{W,p,\delta,T}$ is already proved. Next choose $s_1 \in [1,\infty]$ such that the equality

$$\frac{\mathfrak{n}\gamma}{p} = \frac{\mathfrak{n}}{s_1} = 1 + \frac{\mathfrak{n}}{p} - (\gamma-1)\delta$$

holds. The bound (3.3), the estimate (2.38) on $\left\|S_{\alpha,\beta}(t)\right\|_{p,p_0}$ and the Hölder inequality give

$$\|u_{m+1}\|_{W,p,\delta,T} \leq C\left(\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} + \varphi(T)\,\|1+|u_m|\|^{\gamma}_{W,p,\delta,T}\right),$$

where

$$\begin{aligned}\varphi(T) &:= \sup_{t\in(0,T]} (W(t))^{\delta} \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1} (W(s))^{-\gamma\delta} ds \\ &= (g(T))^{\gamma-1} \sup_{t\in(0,T]} (t\wedge 1)^{\delta} \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|^{-\gamma\delta}_{p,s_1} (s\wedge 1)^{-\delta\gamma}\, ds. \end{aligned} \tag{3.17}$$

The second factor on the right-hand side of (3.17) can be estimated by application of Lemma A with the choices $p := p$, $s_1 := s_1$, $s_2 := \infty$, $\theta := \gamma\delta$, $\delta := \delta$ and $\varepsilon := 0$ (as well as $f(s) \equiv 1$). Then one has $\varphi(T) \leq C(g(T))^{\gamma-1}$ and $\lim_{T\to 0}\varphi(T) = 0$. Henceforth, $u_{m+1} \in E_{W,p,\delta,T}$, for all $m \in \mathbb{N}$ and the claim is proved. The rest of the proof goes exactly as in the proof of Lemma 3.4. We briefly mention the (modified) estimates without giving the full details. In view of the Lipschitz condition (3.4) and the Hölder inequality, we get

$$||u_{m+1} - u_m||_{W,p,\delta,T} \leq \varphi(T)\,||1 + |u_m| + |u_{m-1}|||^{\gamma-1}_{W,p,\delta,T}\,||u_m - u_{m-1}||_{W,p,\delta,T},$$

for all $m \geq 2$. As usual, defining $U := ||u_1||_{W,p,\delta,T} + 2\,||u_2 - u_1||_{W,p,\delta,T}$ and choosing a small enough time $T_* \in (0,1]$ such that $C\varphi(T_*)\,(1+2U)^{\gamma-1} \leq 1/2$, we obtain the analogue of (3.11) in the space E_{W,p,δ,T_*} instead of E_{p,δ,T_*}. Therefore, we deduce again the existence of a limit $u \in E_{W,p,\delta,T_*}$ such that

$$\lim_{m\to\infty} ||u_m - u||_{W,p,\delta,T_*} = 0.$$

The estimates (3.13) and (3.14) concerning the nonlinearity F are proved exactly as in Lemma 3.4. The estimate (3.15) concerning the initial datum u_0 reads

$$\left\|u(\cdot,t) - S_{\alpha,\beta}(t)u_0\right\|_{L^{p_0}(\Omega\setminus\Sigma)} \leq C(g(t))^{\gamma}\,||1+|u|||^{\gamma}_{W,p,\delta,T_*}, \tag{3.18}$$

for all $t \in [0,T_*]$. We now recall that $\lim_{t\to 0} g(t) = 0$ by Lemma B (b). Thus, u is a mild solution of (3.1) in the sense of Definition 3.1 on the time interval $[0,T_*]$. The uniqueness of the mild solution follows from an argument that is similar to the computation (3.16); we omit the details. □

We conclude the proof of Theorem 3.3 by verifying the following statement.

Lemma 3.6. *Assume* **(F3)** *and* **(F4)** *and let $u_0 \in X^{\infty}(\Omega\setminus\Sigma)$. Then the assertion of Theorem 3.3 is satisfied.*

Proof. In this case $\delta = 0$ and $\varepsilon := 1$. The proof exploits a Picard iteration argument for the sequence (3.6) in the space $E_{\infty,0,T}$. Define a measurable function Q such that $Q(|u|) = Q_+(|u_+|)$ whenever $u = u_+$ (a.e.) on Ω_+, and $Q(|u|) = Q_-(|u_-|)$ when $u = u_-$ (a.e.) on Ω_-. The conditions **(F3)** and **(F4)** on the nonlinearity F then read

$$|F(u)| \leq Q(|u|)\,, \text{ a.e. in } \Omega\setminus\Sigma, \tag{3.19}$$

and

$$|F(u) - F(v)| \leq Q(|u| + |v|)\,|u - v|\,, \text{ a.e. on } \Omega\setminus\Sigma. \tag{3.20}$$

We refrain from giving the entire details but briefly mention the main estimates. In a similar fashion to the proof of Lemma 3.4, with some minor (inessential) modifications we obtain

$$\begin{cases} |||u_{m+1}|||_{\infty,0,T} \leq C\left(\|u_0\|_{L^\infty(\Omega\backslash\Sigma)} + TQ\left(|||u_m|||_{\infty,0,T}\right)\right), \\ |||u_{m+1} - u_m|||_{\infty,0,T} \leq TQ\left(|||u_m|||_{\infty,0,T} + |||u_{m-1}|||_{\infty,0,T}\right) |||u_m - u_{m-1}|||_{\infty,0,T}, \\ \left\|u(\cdot,t) - S_{\alpha,\beta}(t)u_0\right\|_{L^\infty(\Omega\backslash\Sigma)} \leq tQ\left(|||u|||_{\infty,0,T}\right), \end{cases} \tag{3.21}$$

for all $t \in [0, T]$, with $T > 0$. We leave the details to the interested reader since the arguments are almost *verbatim* to those performed in the proof of Lemma 3.4. The proof is finished. □

We also give the following continuation theorem and conclude that the mild solution is locally bounded in time in the space $L^\infty(\Omega\backslash\Sigma)$. Since the corresponding arguments are somewhat standard (see [44]) we mention only some brief details.

Theorem 3.7 (Continuation and local boundedness). *Let the assumptions of Theorem 3.3 be satisfied. Then the mild solution of problem (3.1) has a maximal time interval of existence $T_{\max} > 0$ and either $T_{\max} = \infty$, or $T_{\max} < \infty$ and*

$$\lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^{p_0}(\Omega\backslash\Sigma)} = \infty. \tag{3.22}$$

In the case (c) of Theorem 3.3, (3.22) only holds under the additional assumption that the set

$$\kappa(\Pi) := \left\{ \frac{u(\cdot,t)}{\|u(\cdot,t)\|_{L^{p_0}(\Omega\backslash\Sigma)}} : u(\cdot,t) \in L^{p_0}(\Omega\backslash\Sigma),\ t \in [0, T_{\max}),\ \|u(\cdot,t)\|_{L^{p_0}(\Omega\backslash\Sigma)} \neq 0 \right\} \tag{3.23}$$

is precompact in $L^{p_0}(\Omega\backslash\Sigma)$. Finally, every mild solution satisfies

$$\sup_{t\in[T_1,T_2]} \|u(\cdot,t)\|_{L^\infty(\Omega\backslash\Sigma)} < \infty, \text{ for all } T_1, T_2 \in (0, T_{\max}). \tag{3.24}$$

Proof. We first claim that

$$\sup_{t\in[T_0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega\backslash\Sigma)} < \infty, \text{ for all } T_0 \in (0, T],$$

where $T > 0$ is the existence time defined in the statement of Theorem 3.3. Obviously, the case $p_0 = \infty$ is already contained in the proof of Theorem 3.3, so it suffices to take $p_0 < \infty$. Consider the two sequences $\{p_i\}$, $\{\delta_i\}$ as constructed by Lemma C (see Appendix) such that $\delta_i \in (0, 1)$, $i = 1, \ldots, k$, and $p_0 < p_1 < \cdots < p_k = \infty$. One can now inductively apply for each $i = 1, \ldots, k$, the statements of Lemma 3.4 (in the case (a)) and Lemma 3.5 (in the case (c)), respectively, to show

$$\sup_{t\in[T_0,T]} \|u(\cdot,t)\|_{L^{p_i}(\Omega\backslash\Sigma)} \leq C_{T_0,T},\ i = 1, \ldots, k. \tag{3.25}$$

The argument leading to (3.25) is contained in the proof of [44, Lemma 13] and follows without any modifications. For the existence of a maximal interval time of existence, as in the statement of Theorem 3.7, as well as the validity of (3.24), the constructive arguments using the mild solution can be found in the proof of [44, Theorems 1 and 2, pp. 51–54]. □

4. Global mild solutions and regularity

We are next concerned with further regularity properties for the mild solution of (3.1) and global well-posedness results under sufficiently general conditions on the nonlinearities $f_{\pm}$.

Definition 4.1. Let $p \in (1, \infty)$. By a *strong solution u of* (3.1) *on the time interval I* we mean
(a) u is a mild solution in the sense of Definition 3.1 (with $p_0 = \infty$).
(b) $u \in C^{0,\tau}(I; L^{\infty}(\Omega \backslash \Sigma))$, for some $\tau \in (0, 1)$.
(c) $u(\cdot, t) \in D(A_{\alpha,\beta,p})$ and $\partial_t u(\cdot, t) \in L^p(\Omega \backslash \Sigma)$ are well-defined for almost all $t \in I \backslash \{0\}$.
(d) $\partial_t u(\cdot, t) + A_{\alpha,\beta,p} u(\cdot, t) = F(u(\cdot, t))$ is satisfied for almost all $t \in I \backslash \{0\}$.

Theorem 4.2 (Strong solutions on $[0, T_{\max})$). *Let $u_0 \in D(A_{\alpha,\beta,p})$ for some $p \in (\mathfrak{n}, \infty)$. There exists a unique strong solution of* (3.1) *in the sense of Definition 4.1 on the time interval* $[0, T_{\max})$ *provided that assumptions **(F3)** and **(F4)** are satisfied. In particular, Eqs.* (1.8)–(1.10) *are satisfied a.e. in $\Omega_+ \times [0, T_{\max})$, in $\Omega_- \times [0, T_{\max})$ and on $\Sigma \times [0, T_{\max})$, respectively.*

Proof. By Lemma 2.17, $u_0 \in D(A_{\alpha,\beta,p}) \subset X^{\infty}(\Omega \backslash \Sigma)$. Hence, by application of Theorem 3.7 there exists a (unique) mild solution $u \in E_{\infty,0,T}$, $T \in (0, T_{\max})$, that is given by an integral solution (see Definition 3.1 (d)). Furthermore, let $T \in (0, T_{\max})$ and recall by (3.24) that

$$\sup_{t \in [0,T]} \|u(\cdot, t)\|_{L^{\infty}(\Omega \backslash \Sigma)} \leq C_T. \tag{4.1}$$

For any mild solution that satisfies (4.1), we define $H : (0, T_{\max}) \to L^{\infty}(\Omega \backslash \Sigma)$, as $H(t) := F(u(x, t))$ where we recall that the locally Lipschitz function F obeys the conditions (3.19) and (3.20). In particular, we have $\sup_{t \in [0,T]} \|H(t)\|_{L^{\infty}(\Omega \backslash \Sigma)} \leq Q(C_T)$, where Q is the same function as in the proof of Lemma 3.6. With this new definition, the integral solution for the mild solution u can be written

$$u(\cdot, t) = S_{\alpha,\beta}(t) u_0 + \int_0^t S_{\alpha,\beta}(t - s) H(s)\, ds, \tag{4.2}$$

for all $t \in (0, T_{\max})$. Let us now check in what sense the initial condition $u(\cdot, 0) = u_0$ is understood. To this end, let $\theta \in (\mathfrak{n}/p, 1)$ (where $p \in (\mathfrak{n}, \infty)$) and observe that the inclusion $D(A^{\theta}_{\alpha,\beta,p}) \subset L^{\infty}(\Omega \backslash \Sigma)$ is continuous. In view of Remark 2.15 we also recall from [25, p. 26] (since the semigroup is analytic) that for all $t > 0$,

$$\left\| A^{-(1-\tau)}_{\alpha,\beta,p} \left(S_{\alpha,\beta}(t) - I \right) \right\|_{p,p} \leq C_p (t \wedge 1)^{1-\tau}, \text{ for all } \tau \in (0, 1) \tag{4.3}$$

and

$$\left\| A^{\tau}_{\alpha,\beta,p} S_{\alpha,\beta}(t) \right\|_{p,p} \leq C_p (t \wedge 1)^{-\tau}, \text{ for all } \tau \in [0, 1]. \tag{4.4}$$

In all the estimates that follow, we let $0 \leq t < t + h \leq T < T_{\max}$. Using (4.2), we estimate

$$\begin{aligned} \left\| A^{\theta}_{\alpha,\beta,p} (u(\cdot, t) - u_0) \right\|_{L^p(\Omega \backslash \Sigma)} &\leq C (t \wedge 1)^{1-\theta} \left\| A_{\alpha,\beta,p} u_0 \right\|_{L^p(\Omega \backslash \Sigma)} \\ &\quad + \int_0^t \left\| A^{\theta}_{\alpha,\beta,p} S_{\alpha,\beta}(t - s) \right\|_{p,p} \|H(s)\|_{L^p(\Omega \backslash \Sigma)}\, ds \\ &\leq C (t \wedge 1)^{1-\theta} \left\| A_{\alpha,\beta,p} u_0 \right\|_{L^p(\Omega \backslash \Sigma)} \end{aligned}$$

$$+ C\int_0^t ((t-s)\wedge 1)^{-\theta}\, ds \sup_{t\in[0,T]} \|H(t)\|_{L^\infty(\Omega\setminus\Sigma)}$$
$$\leq C(t\wedge 1)^{1-\theta}\left[\left\|A_{\alpha,\beta,p}u_0\right\|_{L^p(\Omega\setminus\Sigma)} + Q(C_T)\right],$$

owing to (4.4). It follows that

$$\|u(\cdot,t)-u_0\|_{L^\infty(\Omega\setminus\Sigma)} \leq C(t\wedge 1)^{1-\theta}\left[\left\|A_{\alpha,\beta,p}u_0\right\|_{L^p(\Omega\setminus\Sigma)} + Q(C_T)\right],$$

for all $t\in[0,T]$. Next, we show that u is continuous in time. The integral solution (4.2) yields

$$u(\cdot,t+h) = S_{\alpha,\beta}(h)\,u(\cdot,t) + \int_0^h S_{\alpha,\beta}(h-s)\,H(t+s)\,ds. \tag{4.5}$$

Our goal is to estimate $u(\cdot,t+h)-u(\cdot,t)$, for $h \ll 1$. Let now $\theta,\eta\in\left(\mathfrak{n}/p,1\right)$ and a sufficiently small $\mu\in(0,1)$ such that $\eta+\mu=\theta$. For the "linear" part of (4.5), owing to (4.3) and (4.2) we have the estimate

$$\begin{aligned}
&\left\|A^{\eta}_{\alpha,\beta,p}\left(S_{\alpha,\beta}(h)\,u(\cdot,t)-u(\cdot,t)\right)\right\|_{L^p(\Omega\setminus\Sigma)}\\
&\quad\leq \left\|A^{-\mu}_{\alpha,\beta,p}\left(S_{\alpha,\beta}(h)-I\right)\right\|_{p,p}\left\|A^{\mu+\eta}_{\alpha,\beta,p}u(\cdot,t)\right\|_{L^p(\Omega\setminus\Sigma)}\\
&\quad\leq C(h\wedge 1)^{\mu}\left(\left\|A^{\theta}_{\alpha,\beta,p}u_0\right\|_{L^p(\Omega\setminus\Sigma)} + \int_0^t \left\|A^{\theta}_{\alpha,\beta,p}S_{\alpha,\beta}(t-s)\right\|_{p,p}\|H(s)\|_{L^p(\Omega\setminus\Sigma)}\,ds\right)\\
&\quad\leq C(h\wedge 1)^{\mu}\left(\|u_0\|_{D(A_{\alpha,\beta,p})} + Q(C_T)\int_0^t ((t-s)\wedge 1)^{-\theta}\,ds\right)\\
&\quad\leq C(h\wedge 1)^{\mu}\left(\|u_0\|_{D(A_{\alpha,\beta,p})} + Q(C_T)(t\wedge 1)^{1-\theta}\right).
\end{aligned}\tag{4.6}$$

Observe preliminarily that this quantity goes to zero as $h\to 0^+$, for all $t\in[0,T]$. Thus, exploiting (4.6) and once again (4.4) we get

$$\begin{aligned}
&\left\|A^{\eta}_{\alpha,\beta,p}\left(u(\cdot,t+h)-u(\cdot,t)\right)\right\|_{L^p(\Omega\setminus\Sigma)}\\
&\quad\leq \left\|A^{\eta}_{\alpha,\beta,p}\left(S_{\alpha,\beta}(h)\,u(\cdot,t)-u(\cdot,t)\right)\right\|_{L^p(\Omega\setminus\Sigma)}\\
&\qquad + \int_0^h \left\|A^{\eta}_{\alpha,\beta,p}S_{\alpha,\beta}(h-s)\right\|_{p,p}\|H(t+s)\|_{L^p(\Omega\setminus\Sigma)}\,ds\\
&\quad\leq C(h\wedge 1)^{\mu}\left(\|u_0\|_{D(A_{\alpha,\beta,p})} + Q(C_T)(t\wedge 1)^{1-\theta}\right)\\
&\qquad + Q(C_T)\int_0^h ((h-s)\wedge 1)^{-\eta-\mu}((h-s)\wedge 1)^{\mu}\,ds\\
&\quad\leq C(h\wedge 1)^{\mu}\left(\|u_0\|_{D(A_{\alpha,\beta,p})} + Q(C_T)(t\wedge 1)^{1-\theta}\right) + Q(C_T)\,T^{\mu}(h\wedge 1)^{1-\theta},
\end{aligned}\tag{4.7}$$

for all $t\in(0,T]$ and $h\in[0,T-t]$. By Lemma 2.17, the inclusion $D(A^{\eta}_{\alpha,\beta,p})\subset L^\infty(\Omega\setminus\Sigma)$ is continuous. Therefore, choosing $\tau=\min\{\mu,1-\theta\}\in(0,1)$, for sufficiently small h, we deduce from (4.7) that

$$\|u(\cdot,t+h)-u(\cdot,t)\|_{L^\infty(\Omega\setminus\Sigma)} \leq C(h\wedge 1)^{\tau}\left[\|u_0\|_{D(A_{\alpha,\beta,p})} + Q(C_T)\left(T^{1-\theta}+T^{\mu}\right)\right]. \tag{4.8}$$

This estimate implies that $u \in C^{0,\tau}([0,T]; L^{\infty}(\Omega\backslash\Sigma))$, which is the desired conclusion (b) of Definition 4.1. Furthermore, the latter also implies that H is Hölder continuous for all $t \in (0, T_{\max})$, owing to the local Lipschitz continuity of F (see (3.20)) and the fact that $u \in E_{\infty,0,T}$ (i.e., $|\|u\||_{\infty,0,T} \le U$), we have that

$$\|H(t) - H(s)\|_{L^{\infty}(\Omega\backslash\Sigma)} \le C_{T,U} |t-s|^{\tau}, \text{ for all } t, s \in [0,T]. \tag{4.9}$$

Hence, in view of (4.9), the formula (4.2) and the application of [25, Lemma 3.2.1 and Theorem 3.2.2] (with the choice $X = L^p(\Omega\backslash\Sigma)$), we can infer the remaining properties (c), (d) of Definition 4.1. We have verified that u is a strong solution in the sense of Definition 4.1 and this concludes the proof of the theorem. □

As a consequence of the proof of Theorem 4.2, we may also infer the following.

Corollary 4.3 (Global regularity of the bounded mild solution). *Let* ***(F3)*** *and* ***(F4)*** *be satisfied and $u_0 \in X^{\infty}(\Omega\backslash\Sigma)$. Consider u to be the corresponding bounded mild solution in the sense of Definition 3.1 on the interval $I = [0,T]$ or $I = [0,\infty)$. Assume*

$$M := \sup_{t\in I} \|u(\cdot,t)\|_{L^{\infty}(\Omega\backslash\Sigma)} < \infty. \tag{4.10}$$

Then for all $T_0 \in I\backslash\{0\}$, u is a strong solution on the time interval $I_0 := [T_0, T]$ (or $I_0 =: [T_0,\infty)$) in the sense of Definition 4.1 (for any $p \in (\mathfrak{n},\infty)$).

Proof. As in the proof of Theorem 4.2 it suffices to show that $u \in C^{0,\tau}([T_0,T]; L^{\infty}(\Omega\backslash\Sigma))$ for any $[T_0,T] \subset I$, for some $\tau \in (0,1)$. To this end, let $p \in (\mathfrak{n},\infty)$ and $\theta, \eta \in (\mathfrak{n}/p, 1)$ such that $\theta = \eta + \mu$ for a sufficiently small $\mu \in (0,1)$. By the formula for the integral solution u, we obtain owing to Hölder's inequality and (4.4),

$$\begin{aligned}
\left\|A^{\theta}_{\alpha,\beta,p} u(\cdot,t)\right\|_{L^p(\Omega\backslash\Sigma)} &\le \left\|A^{\theta}_{\alpha,\beta,p} S_{\alpha,\beta}(t)\right\|_{p,p} \|u_0\|_{L^{\infty}(\Omega\backslash\Sigma)} \\
&\quad + Q(M) \int_0^t \left\|A^{\theta}_{\alpha,\beta,p} S_{\alpha,\beta}(t-s)\right\|_{p,p} ds \\
&\le C(t\wedge 1)^{-\theta} \|u_0\|_{L^{\infty}(\Omega\backslash\Sigma)} + CQ(M)(t\wedge 1)^{1-\theta} \\
&\le C(T_0 \wedge 1)^{-\theta} \left(\|u_0\|_{L^{\infty}(\Omega\backslash\Sigma)} + Q(M)\right),
\end{aligned} \tag{4.11}$$

for all $t \in [T_0, T]$. Consider now the integral formula (4.5), which for all $T_0 \le t < t+h \le T$, reads

$$u(\cdot, t+h) = S_{\alpha,\beta}(h) u(\cdot,t) + \int_0^h S_{\alpha,\beta}(h-s) F(u(\cdot,t+s)) ds. \tag{4.12}$$

By (4.3) and (4.4), exploiting the bound (4.11) we once again have

$$\begin{aligned}
&\left\|A^{\eta}_{\alpha,\beta,p}(u(\cdot,t+h) - u(\cdot,t))\right\|_{L^p(\Omega\backslash\Sigma)} \\
&\le \left\|A^{\eta}_{\alpha,\beta,p}(S(h) - I) u(\cdot,t)\right\|_{L^p(\Omega\backslash\Sigma)} \\
&\quad + \int_0^h \left\|A^{\eta}_{\alpha,\beta,p} S_{\alpha,\beta}(h-s)\right\|_{p,p} \|F(u(\cdot,t+s))\|_{L^p(\Omega\backslash\Sigma)} ds
\end{aligned}$$

$$\leq \left\| A^{-\mu}_{\alpha,\beta,p} \left(S\left(h\right) - I\right) \right\|_{p,p} \left\| A^{\eta+\mu}_{\alpha,\beta,p} u\left(\cdot,t\right) \right\|_{L^p(\Omega\setminus\Sigma)} + Q\left(M\right) \int_0^h C\left(\left(h-s\right)\wedge 1\right)^{-\eta} ds$$
$$\leq C\left(h\wedge 1\right)^{\mu} \left\| A^{\theta}_{\alpha,\beta,p} u\left(\cdot,t\right) \right\|_{L^p(\Omega\setminus\Sigma)} + C\left(h\wedge 1\right)^{1-\eta} Q\left(M\right)$$
$$\leq C\left(h\wedge 1\right)^{\tau} \left[Q\left(M\right) + \left(T_0 \wedge 1\right)^{-\theta} \left(\|u_0\|_{L^\infty(\Omega\setminus\Sigma)} + Q\left(M\right)\right)\right]. \tag{4.13}$$

Here, we have also set $\tau = \min\{\mu, 1-\eta\} \in (0,1)$. The embedding $D(A^{\eta}_{\alpha,\beta,p}) \hookrightarrow L^\infty(\Omega\setminus\Sigma)$ yields from estimate (4.13) the desired claim that u is τ-Hölder continuous with respect to the $L^\infty(\Omega\setminus\Sigma)$-norm. Thus we may conclude the thesis using the same argument employed at the end of the proof of Theorem 4.2, on any time interval $[T_0, T] \subset I$. The proof is finished. □

Our final goal in this section is to derive an *explicit* uniform L^∞-estimate (i.e., (4.10)) from some given L^r-estimate of the mild solution. In what follows we shall implicitly make use of the fact that every mild solution constructed in Section 3 is in fact a differentiable solution on some maximal interval of existence, satisfying Definition 4.1(d). That is, indeed it is a strong solution in the sense of Definition 4.1. In the case $u_0 \in X^\infty(\Omega\setminus\Sigma)$, this statement is already a consequence of Corollary 4.3 and the local boundedness of the mild solution (see Theorem 3.7). In the case when $u_0 \in L^{p_0}(\Omega\setminus\Sigma)$, $p_0 \in [1,\infty)$, the arguments below can still be made rigorous by employing a regularization procedure in which $u_{0m} \in D(A_{\alpha,\beta,p}) \subset X^\infty(\Omega\setminus\Sigma) \subset L^\infty(\Omega\setminus\Sigma)$, for some $p \geq p_0$, $p \in (\mathfrak{n},\infty)$ such that $u_{0m} \to u_0$ in $L^{p_0}(\Omega\setminus\Sigma)$ as $m \to \infty$ (since $D\left(A_{\alpha,\beta,p}\right)$ is dense in $L^{p_0}(\Omega\setminus\Sigma)$). This is no serious drawback since the corresponding mild solutions u_m associated with the initial datum u_{0m} are indeed strong solutions and every mild solution associated with the initial datum $u_0 \in L^{p_0}(\Omega\setminus\Sigma)$ is locally bounded (see again (3.24)).

One method to derive the uniform a priori L^r-L^∞ bound is to employ the extended Moser–Alikakos scheme that was developed for parabolic problems with fractional diffusion in [15, 17] (cf. also [14, 16] for problems with a standard diffusion mechanism in bounded domains with rough boundaries). This procedure cuts off the use of the Gagliardo–Nirenberg (interpolation) inequality and is based on a crucial lemma (see [15, Lemma 3.4]), and a simple embedding result associated with the linear operator $A_{\alpha,\beta}$ (in our case, see (2.9)). We recall that such a procedure usually requires to test the parabolic equation with powers (of the form $|u|^{l-1} u$, $l \geq 2$) of the solution and is generally quite complicated. Unfortunately, this method presents an important drawback for our nonlocal transmission problem (1.8)–(1.12). It turns out that in this case $|u|^{l-2} u$, $l \geq 2$, need *not* even belong to $\mathbb{W}^{(\alpha,\beta),2}(\Omega\setminus\Sigma)$ (see (2.8)), for a general (smooth) function w unless $w \equiv 1$. We state the following theorem without a proof. However, we remark that it can be obtained by performing some simple modifications of the proof of [15, Theorem 3.7] (cf. also [17]).

Theorem 4.4 (Global a priori estimate: the special case $w \equiv 1$). *Assume that $w \equiv 1$ and $f_\pm$ satisfies*

$$f_\pm\left(\xi\right)\xi \geq -C_{f_\pm}\left(\xi^2+1\right), \text{ for all } \xi \in \mathbb{R}, \text{ for some } C_{f_\pm} > 0.$$

Then every bounded mild solution u satisfies the estimate

$$\sup_{t\in[0,T]} \|u\left(\cdot,t\right)\|_{L^\infty(\Omega\setminus\Sigma)} \leq C\left(\|u_0\|_{L^\infty(\Omega\setminus\Sigma)} + \sup_{t\in[0,T]} \|u\left(\cdot,t\right)\|_{L^1(\Omega\setminus\Sigma)}\right),$$

for some constant $C > 0$ independent of t, T, u_0 and u.

Instead, we shall appeal to a method exploited and further developed in [44] for classical systems of parabolic equations with standard diffusion Δ, and which is based on "*feedback*" and some *bootstrap* arguments. The advantage of the "feedback" argument is that it also extends to the case $\alpha \geq \beta$ and $w \not\equiv 1$ (for a generically smooth w), and uses only elementary inequalities. The next theorem generalizes the "feedback" argument to nonlocal problems with fractional diffusion, including ours (1.8)–(1.12), for the mild solutions associated with an initial datum $u_0 \in X^{p_0}(\Omega\backslash\Sigma)$, $p_0 \in [1,\infty]$, as constructed in Section 3. As before, we recall that

$$\mathfrak{n} = \frac{N}{2\beta} \quad \text{and} \quad \delta = \frac{\mathfrak{n}}{p_0} - \frac{\mathfrak{n}}{p}, \text{ for all } p \in \left[p_0, \infty\right].$$

In what follows, we allow for any (sufficiently) smooth w satisfying the hypothesis of Proposition 2.4.

Theorem 4.5 (Global a priori estimate: the general case $\alpha \geq \beta$). *Let $r_1, r_2 \in (0,\infty]$, $\gamma \in [1,\infty)$ satisfy*

$$(\gamma - 1)\left(\frac{\mathfrak{n}}{r_1} + \frac{1}{r_2}\right) < 1 \text{ and } \frac{\gamma - 1}{r_1} < 1, \text{ if } r_1 < \infty. \tag{4.14}$$

*Assume $f_\pm$ obey the conditions **(F1)** and **(F2)** for some $\gamma \in [1,\infty)$ satisfying (4.14). Let now $u_0 \in X^{p_0}(\Omega\backslash\Sigma)$ ($p_0 \in [1,\infty]$) for which the corresponding mild solution satisfies*

$$\|u\|_{r_1,r_2,T} \leq L(\|u_0\|_{X^{p_0}(\Omega\backslash\Sigma)}) \tag{4.15}$$

on any time interval $[0,T]$, for some positive increasing function $L > 0$ (independent of u, u_0) but which depends on the $X^{p_0}(\Omega\backslash\Sigma)$-norm of u_0. Then problem (3.1) has a unique global mild solution on $[0,\infty)$ in the sense of Definition 3.1. In particular, there exist numbers $\rho > 0$ and $\varepsilon \in (0,1)$ such that the mild solution u satisfies the estimates:

$$\sup_{t\in(0,\infty)} (t \wedge 1)^{\delta} \|u(\cdot,t)\|_{L^p(\Omega\backslash\Sigma)} < \infty, \text{ for all } p \in \left[p_0, \infty\right] \tag{4.16}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega\backslash\Sigma)} \leq C(t\wedge 1)^{-\frac{\mathfrak{n}}{p_0}} \left[\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + (t\wedge 1)^{\varepsilon}\left(\Phi + \Phi^{\rho}\right)\right], \tag{4.17}$$

where $\Phi = \Phi\left(\|u_0\|_{X^{p_0}(\Omega\backslash\Sigma)}\right) := \left(1 + L\left(\|u_0\|_{X^{p_0}(\Omega\backslash\Sigma)}\right)\right)$. Estimate (4.17) holds with $\rho = \gamma$ if one assumes that

$$(\gamma - 1)\left(\frac{\mathfrak{n}}{r_1} + \frac{1}{r_2}\right) < 1 \text{ and } \frac{\gamma}{r_1} \leq 1.$$

The proof of this theorem is based on some subsequent lemmas.

Lemma 4.6. *Let $p_0 \in [1,\infty]$, $r_1, r_2 \in (0,\infty]$, $\gamma \in [1,\infty)$, $b \in [0,1]$, $\varepsilon \in (0,1)$ and $p \in \left[p_0,\infty\right]$ such that*

$$\begin{cases} \gamma(1-b)\left(\dfrac{\mathfrak{n}}{r_1} + \dfrac{1}{r_2}\right) < 1 + (1-\gamma b)\dfrac{\mathfrak{n}}{p} - \varepsilon, \\ \gamma\dfrac{(1-b)}{r_1} + \gamma\dfrac{b}{p} \leq 1, \\ \gamma\dfrac{(1-b)}{r_2} + \gamma b\delta < 1 - \varepsilon, \\ \gamma b < 1, \ \delta = \dfrac{\mathfrak{n}}{p_0} - \dfrac{\mathfrak{n}}{p}. \end{cases} \tag{4.18}$$

*Let $f_\pm$ obey the condition **(F1)** for some $\gamma \in [1,\infty)$ satisfying (4.18). Let the mild solution u for problem (3.1) with an initial datum $u_0 \in X^{p_0}(\Omega\backslash\Sigma)$ satisfy the a priori estimate $\|u\|_{r_1,r_2,T} < \infty$, for any $T > 0$. Furthermore assume $|||u|||_{p,\delta,T} < \infty$ and for $b > 0$ define $U := C_F\left(\|1+|u|\|_{r_1,r_2,T}^{\gamma(1-b)}\right)$. Then there exists a constant $C_* > 0$ independent of u_0, u, U, t and T such that*

$$|||u|||_{p,\delta,T} \leq C_*\left[\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + (T\wedge 1)^\varepsilon\left(U + U^{\frac{1}{1-\gamma b}}\right)\right]. \tag{4.19}$$

Proof. We shall exploit again the integral formulation for the mild solution (see Definition 3.1). By (4.18), there exist $s_1, s_2 \in [1,\infty]$ such that

$$\frac{\mathfrak{n}}{s_1} + \frac{1}{s_2} \leq 1 + \frac{\mathfrak{n}}{p} - \varepsilon, \tag{4.20}$$

$$\frac{1}{s_2} + \gamma b\delta < 1 - \varepsilon, \tag{4.21}$$

$$\gamma\frac{1-b}{r_1} + \frac{\gamma b}{p} \leq \frac{1}{s_1}, \tag{4.22}$$

$$\gamma\frac{1-b}{r_2} \leq \frac{1}{s_2}. \tag{4.23}$$

We have

$$\|u(\cdot,t)\|_{L^p(\Omega\backslash\Sigma)} \leq \left\|S_{\alpha,\beta}(t)\right\|_{p,p_0}\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + \int_0^t\left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1}\|F(u(\cdot,s))\|_{L^{s_1}(\Omega\backslash\Sigma)}\,ds.$$

We use (3.3), to split the nonlinear term into several terms. First, by (4.22) and the Hölder inequality we get for all $t > 0$,

$$\begin{aligned}
&(t\wedge 1)^\delta\|u(\cdot,t)\|_{L^p(\Omega\backslash\Sigma)} \\
&\quad\leq (t\wedge 1)^\delta\left\|S_{\alpha,\beta}(t)\right\|_{p,p_0}\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} \\
&\qquad + (t\wedge 1)^\delta C_F\int_0^t\left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1}\|1+|u(s)|\|_{L^{r_1}(\Omega\backslash\Sigma)}^{\gamma(1-b)}(s\wedge 1)^{-\gamma b\delta}\,ds \\
&\qquad\qquad \times\left(|||1+|u||||_{p,\delta,T}^{\gamma b}\right).
\end{aligned} \tag{4.24}$$

The first summand on the right-hand side of (4.24) can be estimated using the ultracontractivity property (2.38) for $S_{\alpha,\beta}(t)$ as a bounded operator from $L^{p_0}(\Omega\backslash\Sigma)$ into $L^p(\Omega\backslash\Sigma)$. For the second summand we apply Lemma A with the choice $p, s_1, s_2, \delta, \varepsilon$ as above, $\theta := \gamma b\delta$ and $f(s) := C_F\|1+|u(\cdot,s)|\|_{L^{r_1}(\Omega\backslash\Sigma)}^{\gamma(1-b)}$. Hence from (4.24), we deduce

$$|||u|||_{p,\delta,T} \leq C\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + C(T\wedge 1)^\varepsilon p_{s_2}(f)\,|||1+|u||||_{p,\delta,T}^{\gamma b}. \tag{4.25}$$

The function $p_{s_2}(f)$ can be estimated from the same Lemma A using the Hölder inequality on account of (4.23). It follows that $p_{s_2}(f) = C_F\left(\|1+|u|\|_{r_1,r_2,T}^{\gamma(1-b)}\right) = U$. Therefore, (4.25) implies that

$$|||u|||_{p,\delta,T} \leq C\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + C(T\wedge 1)^\varepsilon U\,|||1+|u||||_{p,\delta,T}^{\gamma b}. \tag{4.26}$$

Observe now that (4.26) is already the assertion (4.19) when $b = 0$. In order to show the estimate in the case when $b > 0$, we apply a "feedback" argument to (4.26) by employing the "feedback" inequality of Lemma E with the following choices

$$y := |||u|||_{p,\delta,T}\,,\; z_0 := C\left(\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} + (T\wedge 1)^{\varepsilon}\, U\right),\; z_1 := C\,(T\wedge 1)^{\varepsilon}\, U$$

with $\sigma := \gamma b < 1$. Indeed, (4.26) yields that $y \le z_0 + z_1 y^{\sigma}$ and therefore, we obtain

$$y \le \frac{z_0}{1-\sigma} + z_1^{\frac{1}{1-\sigma}}.$$

The foregoing inequality yields (4.19) with constant $C_* = C/(1-\gamma b) + C^{1/(1-\gamma b)}$.

Next, we can also check in what sense the initial datum is satisfied. By the integral formula and the bound (3.3) we have

$$\begin{aligned}&\left\|u(\cdot,t) - S_{\alpha,\beta}(t)\,u_0\right\|_{L^{p_0}(\Omega\setminus\Sigma)}\\ &\quad\le C_F \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p_0,s_1} \|1+|u(\cdot,s)|\|_{L^{r_1}(\Omega\setminus\Sigma)}^{\gamma(1-b)} \|1+|u(\cdot,s)|\|_{L^{r_1}(\Omega\setminus\Sigma)}^{\gamma b}\, ds\end{aligned}$$

on which we can once again apply Lemma A with the same s_1, s_2 and choice of function f as above, and $\delta := 0$, $p := p_0$, $\theta := \gamma b\delta$ and $\varepsilon := 0$. By (4.20) and (4.21), we can easily verify that the assumptions of Lemma A are indeed verified. We get

$$\begin{aligned}\left\|u(\cdot,t) - S_{\alpha,\beta}(t)\,u_0\right\|_{L^{p_0}(\Omega\setminus\Sigma)} &\le C\,(t\wedge 1)^{\varepsilon}\, p_{s_2}(f)\; |||1+|u|||\,|_{p,\delta,T}^{\gamma b}\\ &\le C\,(t\wedge 1)^{\varepsilon}\, U\, |||1+|u|||\,|_{p,\delta,T}^{\gamma b}, \end{aligned}\tag{4.27}$$

for all $t \in (0,T]$. Finally, it is also easy to check that $\|F(u)\|_{1,1,T} < \infty$, for any $T > 0$ for which u satisfies (4.19). The proof is finished. □

Lemma 4.7. *Let $p_0 \in [1,\infty]$, $r_1, r_2 \in (0,\infty]$, $\gamma \in [1,\infty)$, $b \in [0,1]$, $\varepsilon \in (0,1)$ satisfy*

$$\gamma\,(1-b)\left(\frac{\mathfrak{n}}{r_1} + \frac{1}{r_2}\right) < 1-\varepsilon, \tag{4.28}$$

$$\gamma\frac{1-b}{r_1} \le 1, \tag{4.29}$$

$$\gamma b < 1. \tag{4.30}$$

*Let $f_\pm$ obey the condition **(F1)** for some $\gamma \in [1,\infty)$ that satisfies (4.28)–(4.30). Let the mild solution u for problem (3.1) with an initial datum $u_0 \in X^{p_0}(\Omega\setminus\Sigma)$ satisfy $U < \infty$ for any $T > 0$, where U is defined in the statement of Lemma 4.6. Furthermore for $b > 0$ assume that $\sup_{t\in(0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega\setminus\Sigma)} < \infty$. Then there exists a constant $C_* > 0$ independent of u_0, u, U, t and T such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega\setminus\Sigma)} \le C_*\,(t\wedge 1)^{-\frac{\mathfrak{n}}{p_0}}\left[\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} + (t\wedge 1)^{\varepsilon}\left(U + U^{\frac{1}{1-\gamma b}}\right)\right], \tag{4.31}$$

for all $t \in (0,T]$.

Remark 4.8. In the case $b > 0$ the a priori information $\sup_{t\in(0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega\setminus\Sigma)} < \infty$ is essential to deduce the explicit estimate (4.31) with a constant independent of time and of any $T > 0$. Otherwise, no conclusion can be drawn from the "feedback" argument. On the other hand since every mild solution of Theorem 3.3 is locally bounded by Theorem 3.7 on $(0, T_{\max})$, we can infer from (4.31) that $T_{\max} = \infty$ for as long as U is finite on any time interval $[0, T]$.

Proof of Lemma 4.7. First, we observe that when $p_0 = \infty$ or $b = 0$, the assumptions (4.18) of Lemma 4.6 are satisfied with $p := \infty$, $\delta := 0$ and r_1, r_2, γ as above in (4.28)–(4.30). In this case, the assertion (4.31) is equivalent to the estimate (4.19) of Lemma 4.6. Thus, we may assume that $p_0 \in [1, \infty)$ and $b \in (0, 1]$. We apply an inductive argument with help from Lemma 4.6. To this end, consider the finite sequences $\{p_i\}$ with $p_0 < p_1 < \cdots < p_k = \infty$, and $\{\delta_i\} \in (0, 1)$ for $i = 1, \ldots, k$ as given by Lemma D. We then apply Lemma 4.6 with the choices $p_0 := p_{i-1}, p := p_i, \delta := \delta_i$, the exponents r_1, r_2, γ, b as above in (4.28)–(4.30), and initial datum $u_0 := u(\cdot, t)$ for arbitrary $t \in (0, T]$. It follows that (4.19) of Lemma 4.6 yields for all $h \in (0, T - t]$, $i = 1, \ldots, k$,

$$|||u(t+h)|||_{p_i,\delta_i,h} \le C_i \left[\|u(\cdot, t)\|_{L^{p_{i-1}}(\Omega\backslash\Sigma)} + (h \wedge 1)^{\varepsilon} \left(U + U^{\frac{1}{1-\gamma b}} \right) \right]. \tag{4.32}$$

The choice $t = ih - h$ in (4.32) then gives

$$(h \wedge 1)^{\delta_i} \|u(\cdot, ih)\|_{L^{p_i}(\Omega\backslash\Sigma)} \le C_i \left[\|u(\cdot, ih - h)\|_{L^{p_{i-1}}(\Omega\backslash\Sigma)} + (h \wedge 1)^{\varepsilon} \left(U + U^{\frac{1}{1-\gamma b}} \right) \right], \tag{4.33}$$

for all $i = 1, \ldots, k$ and $h \in (0, T/i]$, for some $C_i < \infty$. An induction argument in (4.33) for $i = 1, \ldots, k$ implies

$$(h \wedge 1)^{\delta_1 + \cdots + \delta_i} \|u(\cdot, ih)\|_{L^{p_i}(\Omega\backslash\Sigma)} \le C_i \left[\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + (h \wedge 1)^{\varepsilon} \left(U + U^{\frac{1}{1-\gamma b}} \right) \right]. \tag{4.34}$$

Since $\delta_i = \mathfrak{n}/p_{i-1} - \mathfrak{n}/p_i$, we readily have $\delta_1 + \cdots + \delta_k = \mathfrak{n}/p_0$ and (4.34) with $i = k$, gives no other than the required estimate (4.31). This completes the proof of the lemma. □

Before we can finish the proof of Theorem 4.5 we also need the following continuous dependence estimate.

Lemma 4.9. *Let $p_0 \in [1, \infty]$, $r_1, r_2 \in (0, \infty]$, $\varepsilon \in (0, 1)$, $p \in [p_0, \infty]$ and assume* **(F2)** *for some $\gamma \in [1, \infty)$ that satisfies*

$$(\gamma - 1) \left(\frac{\mathfrak{n}}{r_1} + \frac{1}{r_2} \right) < 1 - \varepsilon, \tag{4.35}$$

$$\frac{\gamma - 1}{r_1} + \frac{1}{p} \le 1, \tag{4.36}$$

$$\frac{\gamma - 1}{r_2} + \delta < 1 - \varepsilon, \tag{4.37}$$

and the a priori estimate (4.15). Let u_i be any two mild solutions in the sense of Definition 3.1 for any two initial data $u_{0i} \in X^{p_0}(\Omega\backslash\Sigma)$, $i = 1, 2$. Then there exists a constant $C > 0$ independent of u_i, t, T and u_{0i}, such that

$$|||u_1 - u_2|||_{p,\delta,T} \le C \|u_{01} - u_{02}\|_{L^{p_0}(\Omega\backslash\Sigma)} + C (T \wedge 1)^{\varepsilon} \|1 + |u_1| + |u_2|\|^{\gamma-1}_{r_1,r_2,T} |||u_1 - u_2|||_{p,\delta,T}, \tag{4.38}$$

for all $t \in (0, T]$.

Proof. The argument follows in a similar fashion to the computation (4.25) and (4.26) using the local Lipschitz condition (3.4). Choose $s_1, s_2 \in [1, \infty]$ such that

$$\frac{\gamma - 1}{r_1} + \frac{1}{p} \le \frac{1}{s_1}, \quad \frac{\gamma - 1}{r_2} \le \frac{1}{s_2}$$

and

$$\frac{\mathfrak{n}}{s_1} + \frac{1}{s_2} - \frac{\mathfrak{n}}{p} + \varepsilon < 1, \quad \frac{1}{s_2} + \delta \le 1 - \varepsilon.$$

By the integral solution representation for each u_i, by the Hölder inequality and (3.4) we have

$$\begin{aligned} &\|(u_1 - u_2)(\cdot, t)\|_{L^p(\Omega\backslash\Sigma)} \\ &\quad \le \left\|S_{\alpha,\beta}(t)\right\|_{p,p_0} \|u_{01} - u_{02}\|_{L^{p_0}(\Omega\backslash\Sigma)} \\ &\qquad + C_F \, |||u_1 - u_2|||_{p,\delta,T} \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|_{p,s_1} \\ &\qquad\qquad \times \|1 + |u_1(\cdot, s)| + |u_2(\cdot, s)|\|_{L^{r_1}(\Omega\backslash\Sigma)}^{\gamma-1} (s \wedge 1)^{-\delta} \, ds. \end{aligned} \tag{4.39}$$

The first term at the right hand side of (4.39) can be estimated as before using the ultracontractivity estimate for $S_{\alpha,\beta}(t)$. For the second summand in (4.39), we apply Lemma A (whose assumptions are satisfied) with the choices

$$f(s) := C_F \|1 + |u_1(\cdot, s)| + |u_2(\cdot, s)|\|_{L^{r_1}(\Omega\backslash\Sigma)}^{\gamma-1}$$

and p, s_1, s_2, ε as above, and $\theta := \delta$. The foregoing inequality then yields

$$\begin{aligned} |||u_1 - u_2|||_{p,\delta,T} &\le C \|u_{01} - u_{02}\|_{L^{p_0}(\Omega\backslash\Sigma)} \\ &\quad + C(T \wedge 1)^{\varepsilon} p_{s_2}(f) \, |||u_1 - u_2|||_{p,\delta,T}. \end{aligned} \tag{4.40}$$

The functional $p_{s_2}(f)$ can be estimated exploiting Lemma A once more to find

$$p_{s_2}(f) \le C_F \|1 + |u_1| + |u_2|\|_{r_1,r_2,T}^{\gamma-1} < \infty,$$

which is finite by virtue of the assumption (4.15). Thus, (4.40) implies the desired assertion (4.38) of Lemma 4.9. □

Proof of Theorem 4.5. Let $u_0 \in X^{p_0}(\Omega\backslash\Sigma)$ and consider a sequence $\{u_{0j}\}_{j\in\mathbb{N}} \subset D(A_{\alpha,\beta,p}) \subset L^{\infty}(\Omega\backslash\Sigma)$ for $p \ge p_0$, $p \in (\mathfrak{n}, \infty)$, such that

$$\lim_{j\to\infty} \left\|u_{0j} - u_0\right\|_{L^{p_0}(\Omega\backslash\Sigma)} = 0 \tag{4.41}$$

(recall that $D(A_{\alpha,\beta,p})$ is dense in $L^{p_0}(\Omega\backslash\Sigma)$). By Theorem 3.3 there exists a unique mild solution u_j for problem (3.1), which is also smooth by Theorem 4.2, on the time interval $[0, T_j)$, where $T_j > 0$ is the maximal existence time. We can show that $T_j = \infty$, for all $j \in \mathbb{N}$. The assumption (4.14) of Theorem 4.5 implies that there exist numbers $\varepsilon \in (0, 1)$ and $b \in [0, 1/\gamma)$ such that the assumptions (4.28)–(4.30) of Lemma 4.7 are satisfied. Then we can infer from the estimate (4.31) that

$$\left\|u_j(\cdot, t)\right\|_{L^{\infty}(\Omega\backslash\Sigma)} \le C_*(t \wedge 1)^{-\frac{\mathfrak{n}}{p_0}} \left[\|u_0\|_{L^{p_0}(\Omega\backslash\Sigma)} + (t \wedge 1)^{\varepsilon}\left(U + U^{\frac{1}{1-\gamma b}}\right)\right], \tag{4.42}$$

for all $j \in \mathbb{N}$, and $t \in (0, T_j)$. The constant $C_* > 0$ is clearly independent of j. The assertion (3.22) together with (4.42) and the fact that

$$U = C_F \sup_{j\in\mathbb{N}} \left\{ \left(\left\| 1 + |u_j| \right\|_{r_1,r_2,T}^{\gamma(1-b)} \right) : T \in (0,\infty) \right\} < \infty$$

uniformly in j, owing to condition (4.15), shows that $T_j = \infty$ for all $j \in \mathbb{N}$.

The final goal of the proof is to show, along a proper subsequence (still denoted by) $\{u_j\}$, that u_j converges to a function u on any interval $(0, T] \subset (0, \infty)$. To this end, we also observe that due to the uniform estimate (4.42) and the assumption (4.15), we have

$$\begin{aligned} V &:= C_F \sup_{j,m\in\mathbb{N}} \left\{ \left(\left\| 1 + |u_j| + |u_m| \right\|_{r_1,r_2,T}^{\gamma-1} \right) : T \in (0,\infty) \right\} \\ &\leq C \left(1 + 2L \left(\|u_0\|_{L^{p_0}(\Omega\setminus\Sigma)} \right) \right)^{\gamma(1-b)}. \end{aligned} \tag{4.43}$$

We choose the initial time ih for an arbitrary $h \in (0, \infty)$ and $i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The continuous dependence estimate (4.38) yields in light of the uniform bound (4.43) that

$$\begin{aligned} &\left|\left|\left| \left(u_j - u_l \right)(ih + \cdot) \right|\right|\right|_{p,\delta,h} \\ &\quad \leq C \left|\left| \left(u_j - u_l \right)(\cdot, ih) \right|\right|_{L^{p_0}(\Omega\setminus\Sigma)} + C (h \wedge 1)^{\varepsilon} V \left|\left|\left| \left(u_j - u_l \right)(ih + \cdot) \right|\right|\right|_{p,\delta,h}^{\gamma b}, \end{aligned} \tag{4.44}$$

for all $j, l \in \mathbb{N}$, and $i \in \mathbb{N}_0$ and $h > 0$. Choosing $h \ll 1$ small enough such that $C (h \wedge 1)^{\varepsilon} V \leq 1/2$, from (4.44) we get for all $i \in \mathbb{N}_0$ that

$$\left|\left|\left| \left(u_j - u_l \right)(ih + \cdot) \right|\right|\right|_{p,\delta,h} \leq C \left|\left| \left(u_j - u_l \right)(\cdot, ih) \right|\right|_{L^{p_0}(\Omega\setminus\Sigma)}.$$

In particular, owing to (4.41) and a continuation argument, we obtain that $\{u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $E_{p,\delta,T}$, for all $T \in (0, \infty)$. Therefore there exists a function $u \in E_{p,\delta,T}$, for any $T \in (0, \infty)$, such that

$$\lim_{j\to\infty} \left|\left|\left| u_j - u \right|\right|\right|_{p,\delta,T} = 0, \text{ for all } T \in (0, \infty). \tag{4.45}$$

Fixing now a time $t \in (0, T] \subset (0, \infty)$, (4.45) also yields that $u_j(x, t) \to u(x, t)$ for almost all $x \in \Omega\setminus\Sigma$; we conclude that (4.42) also holds for $u(\cdot, t)$ (as well as the estimate (4.16) is verified). Thus, u is well-defined globally on $(0, \infty)$. In order to show that the limit solution u is also a mild solution in the sense of Definition 3.1, we argue exactly as in the proof of Theorem 3.1, by taking advantage of the strong convergence (4.45) to pass to the limit in the integral solution representation for u_j. We leave the (repetitive) details to the interested reader. The proof is finished. □

We conclude this section with the following.

Corollary 4.10. *Under the assumptions of Theorem 3.1 and Theorem 4.5, every mild* (globally-defined) *solution of problem* (3.1) *with initial datum* $u_0 \in X^{p_0}(\Omega\setminus\Sigma)$, $p_0 \in [1, \infty]$, *is also a strong solution on* $[\delta, \infty)$, *for any* $\delta > 0$ *in the sense of Definition 4.1.*

5. The case $\alpha \leq \beta$ and concluding remarks

In this article, we have considered a general family of transmission problems with anomalous diffusion that has not been considered or analyzed anywhere in the literature before. We have given a unified analysis of our transmission problem in the case $\alpha \geq \beta$ using basic tools in nonlinear analysis and Sobolev function theory together with semigroup methods. We have developed well-posedness results for our family of transmission models which include local existence results (Section 3) and global regularity results (Section 4).

5.1. The case $\alpha \leq \beta$

The present analysis can be extended to nonlocal transmission problems with fractional diffusion in the case $1/2 < \alpha \leq \beta < 1$. To this end, we briefly verify that the parabolic system (1.8) and (1.9), (1.11) subject to the transmission conditions

$$\widetilde{w}u_+ = u_-, \quad \widetilde{\mathcal{N}}_{\widetilde{w}}(u_+, u_-) + \mathfrak{b}u_+ = 0, \quad \text{on } (0,T) \times \Sigma, \tag{5.1}$$

and initial condition (1.12) is also *well-posed* (see Figure 1). Here, we have defined the jump

$$\widetilde{\mathcal{N}}_{\widetilde{w}}(u_+(x), u_-(x)) := C_\beta \mathcal{N}^{2-2\beta} u_+(x) - C_\alpha \widetilde{w}(x) \mathcal{N}^{2-2\alpha} u_-(x), \tag{5.2}$$

where $\widetilde{w} \in W^{\theta,2}(\Sigma)$ with $\theta \geq \alpha$ such that $\theta + \beta - \alpha > (N-1)/2$. Any $\theta > N/2$ works. We consider the fractional order Sobolev space

$$\widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) := \left\{ u \in L^2(\Omega\backslash\Sigma) : u_- \in W^{\alpha,2}(\Omega_-),\ u_+ \in W^{\beta,2}(\Omega_+) \text{ and } u_- = \widetilde{w}u_+ \text{ on } \Sigma \right\},$$

endowed with the norm defined by

$$\|u\|^2_{\widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)} = \int_{\Omega\backslash\Sigma} |u|^2 dx + \int_{\Omega_-}\int_{\Omega_-} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{N+2\alpha}} dxdy + \int_{\Omega_+}\int_{\Omega_+} \frac{|u_+(x) - u_+(y)|^2}{|x-y|^{N+2\beta}} dxdy.$$

We notice that (2.10) and (2.11) remain true with $\mathbb{W}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$ replaced by $\widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$. The corresponding embedding (2.9) becomes

$$\widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma) \hookrightarrow L^{\frac{2N}{N-2\alpha}}(\Omega\backslash\Sigma). \tag{5.3}$$

In addition we have that

$$\|u\|^2_* := \int_\Sigma |u_+|^2 d\sigma + \int_{\Omega_-}\int_{\Omega_-} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{N+2\alpha}} dxdy + \int_{\Omega_+}\int_{\Omega_+} \frac{|u_+(x) - u_+(y)|^2}{|x-y|^{N+2\beta}} dxdy, \tag{5.4}$$

defines an equivalent norm on $\widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$. Its proof follows the lines of the proof of Lemma 2.5.

Let $\mathfrak{b} \in L^\infty(\Sigma)$ be nonnegative. We define the bilinear symmetric form $\widetilde{\mathcal{E}}_{\alpha,\beta}$ with domain $D(\widetilde{\mathcal{E}}_{\alpha,\beta}) := \widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma)$ by

$$\widetilde{\mathcal{E}}_{\alpha,\beta}(u,v) = \frac{C_{N,\alpha}}{2}\int_{\Omega_-}\int_{\Omega_-} \frac{(u_-(x) - u_-(y))(v_-(x) - v_-(y))}{|x-y|^{N+2\alpha}} dxdy + \frac{C_{N,\beta}}{2}\int_{\Omega_+}\int_{\Omega_+} \frac{(u_+(x) - u_+(y))(v_+(x) - v_+(y))}{|x-y|^{N+2\beta}} dxdy + \int_\Sigma \mathfrak{b}u_+v_+ d\sigma. \tag{5.5}$$

Proceeding exactly as in Proposition 2.11, we infer that $\widetilde{\mathcal{E}}_{\alpha,\beta}$ is a Dirichlet form; more precisely, it is closed and Markovian on $L^2(\Omega\backslash\Sigma)$. Let now $\widetilde{A}_{\alpha,\beta}$ be the self-adjoint operator on $L^2(\Omega\backslash\Sigma)$ associated with $\widetilde{\mathcal{E}}_{\alpha,\beta}$. It has the following properties (see the proofs of Proposition 2.12 and Theorem 2.16, and (5.3)).

Theorem 5.1. *The operator $\widetilde{A}_{\alpha,\beta}$ is given by*

$$D(\widetilde{A}_{\alpha,\beta}) = \Big\{u \in \widetilde{\mathbb{W}}^{(\alpha,\beta),2}(\Omega\backslash\Sigma),\ (-\Delta)^{\alpha}_{\Omega_-} u_- \in L^2(\Omega_-),\ (-\Delta)^{\beta}_{\Omega_+} u_+ \in L^2(\Omega_+),$$
$$\mathcal{N}^{2-2\alpha} u_- = 0 \ \text{ on } \ \Gamma,\ \widetilde{\mathcal{N}}_{\widetilde{w}}(u_+, u_-) + \mathfrak{b} u_+ = 0 \ \text{ on } \ \Sigma\Big\} \tag{5.6}$$

and, for $u \in D(\widetilde{A}_{\alpha,\beta})$, we have

$$\widetilde{A}_{\alpha,\beta} u = (-\Delta)^{\beta}_{\Omega_+} u_+ \text{ on } \Omega_+, \text{ and } \widetilde{A}_{\alpha,\beta} u = (-\Delta)^{\alpha}_{\Omega_-} u_- \text{ on } \Omega_-. \tag{5.7}$$

Finally, the following are also true.

(a) *$\widetilde{A}_{\alpha,\beta}$ has a compact resolvent, and hence, has a discrete spectrum. The spectrum of $\widetilde{A}_{\alpha,\beta}$ is an increasing nonnegative sequence of real numbers $\{\widetilde{\lambda}_k\}_{k\in\mathbb{N}}$ such that $\widetilde{\lambda}_k \to \infty$. If $\mathfrak{b}$ satisfies (2.22), then $\widetilde{\lambda}_1 > 0$. If $\mathfrak{b} = 0$ σ-a.e. on Σ, then $\widetilde{\lambda}_1 = 0$. If $\widetilde{u}_k$ is an eigenfunction of $\widetilde{A}_{\alpha,\beta}$ associated with the eigenvalue $\widetilde{\lambda}_k$, then $\widetilde{u}_k \in D(\widetilde{A}_{\alpha,\beta}) \cap L^\infty(\Omega\backslash\Sigma)$.*

(b) *The operator $-\widetilde{A}_{\alpha,\beta}$ generates a submarkovian semigroup $(e^{-t\widetilde{A}_{\alpha,\beta}})_{t\geq 0}$ on $L^2(\Omega\backslash\Sigma)$. The latter can be also extended to a contraction (compact) semigroup $\widetilde{S}_{\alpha,\beta,p}(t) := e^{-t\widetilde{A}_{\alpha,\beta,p}}$ on $L^p(\Omega\backslash\Sigma)$ for every $p \in [1,\infty]$, and each semigroup is strongly continuous if $p \in [1,\infty)$ and bounded analytic if $p \in (1,\infty)$.*

(c) *For every $1 \leq q \leq p \leq \infty$, there exists a constant $C > 0$ such that for every $f \in L^q(\Omega\backslash\Sigma)$ and $t > 0$,*

$$\|e^{-t\widetilde{A}_{\alpha,\beta}} f\|_{L^p(\Omega\backslash\Sigma)} \leq C\,(t \wedge 1)^{-\frac{N}{2\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_{L^q(\Omega\backslash\Sigma)}.$$

On account of Theorem 5.1, all the results on well-posedness and global regularity of mild solutions for problem (1.8) and (1.9), (1.11), (5.1), as stated in Sections 3–4, remain valid with one simple modification: one redefines $\mathfrak{n} := N/(2\alpha)$ (instead of $\mathfrak{n} = N/(2\beta)$). To avoid redundancy, we refrain from explicitly stating these results and their verbatim proofs. Finally, we also recover the result of Theorem 2.22 with the following modification: in the cases (a) and (b) assume $q > N/(2\alpha)$.

5.2. Related local-nonlocal transmission problems

Although only the case of fully nonlocal transmission problems of the form (1.8) and (1.9) have been considered in the paper, the corresponding methods are of general nature and can be successfully applied to other related transmission problems. Some of them are briefly discussed below. A more detailed exposition of proofs and similar problems will be given elsewhere. In particular, all the results obtained in the paper are also valid for local-nonlocal transmission problems in the following cases: $1/2 < \beta < \alpha = 1$ or $1/2 < \alpha < \beta = 1$. If

$1/2 < \beta < \alpha = 1$, then the local-nonlocal transmission problem (1.8)–(1.12) reads

$$\begin{cases} \partial_t u_+ + (-\Delta)^{\beta}_{\Omega_+} u_+ + f_+(u_+) = 0, & \text{in } J \times \Omega_+, \\ \partial_t u_- - \Delta u_- + f_-(u_-) = 0 & \text{in } J \times \Omega_-, \\ u_+ = w u_-, & \text{on } J \times \Sigma \\ C_\beta w \mathcal{N}^{2-2\beta} u_+ - \partial_\nu u_- + \mathfrak{b} u_- = 0 & \text{on } J \times \Sigma \\ \partial_\nu u_- = 0, & \text{on } J \times \Gamma, \\ u_+(\cdot, 0) = u_+^0, & \text{in } \Omega_+, \\ u_-(\cdot, 0) = u_-^0, & \text{in } \Omega_-. \end{cases}$$

The associated bilinear form $\mathcal{E}_{1,\beta}$ is given for $u, v \in D(\mathcal{E}_{1,\beta}) = \mathbb{W}^{(1,\beta),2}(\Omega \backslash \Sigma)$ by

$$\mathcal{E}_{1,\beta}(u,v) = \int_{\Omega_-} \nabla u_- \cdot \nabla v_- dx + \frac{C_{N,\beta}}{2} \int_{\Omega_+} \int_{\Omega_+} \frac{(u_+(x) - u_+(y))(v_+(x) - v_+(y))}{|x-y|^{N+2\beta}} dxdy + \int_\Sigma \mathfrak{b} u_- v_- d\sigma,$$

where

$$\mathbb{W}^{(1,\beta),2}(\Omega \backslash \Sigma) := \Big\{ u \in L^2(\Omega \backslash \Sigma) : u_- \in W^{1,2}(\Omega_-),\ u_+ \in W^{\beta,2}(\Omega_+) \text{ and } u_+ = w u_- \text{ on } \Sigma \Big\}.$$

In the case $1/2 < \alpha < \beta = 1$, the local-nonlocal transmission problem (1.8)–(1.12) corresponds to

$$\begin{cases} \partial_t u_+ - \Delta u_+ + f_+(u_+) = 0 & \text{in } J \times \Omega_+, \\ \partial_t u_- + (-\Delta)^{\alpha}_{\Omega_-} u_- + f_-(u_-) = 0, & \text{in } J \times \Omega_-, \\ \widetilde{w} u_+ = u_-, & \text{on } J \times \Sigma \\ \partial_\nu u_+ - C_\alpha \widetilde{w} \mathcal{N}^{2-2\alpha} u_- + \mathfrak{b} u_+ = 0 & \text{on } J \times \Sigma \\ \mathcal{N}^{2-2\alpha} u_- = 0, & \text{on } J \times \Gamma, \\ u_+(\cdot, 0) = u_+^0, & \text{in } \Omega_+, \\ u_-(\cdot, 0) = u_-^0, & \text{in } \Omega_-. \end{cases}$$

The associated bilinear form $\widetilde{\mathcal{E}}_{\alpha,1}$ is given for $u, v \in D(\widetilde{\mathcal{E}}_{\alpha,1}) := \widetilde{\mathbb{W}}^{(\alpha,1),2}(\Omega \backslash \Sigma)$ by

$$\widetilde{\mathcal{E}}_{\alpha,1}(u,v) = \frac{C_{N,\alpha}}{2} \int_{\Omega_-} \int_{\Omega_-} \frac{(u_-(x) - u_-(y))(v_-(x) - v_-(y))}{|x-y|^{N+2\alpha}} dxdy + \int_{\Omega_+} \nabla u_+ \cdot \nabla v_+ dx + \int_\Sigma \mathfrak{b} u_+ v_+ d\sigma,$$

where

$$\widetilde{\mathbb{W}}^{(\alpha,1),2}(\Omega \backslash \Sigma) := \Big\{ u \in L^2(\Omega \backslash \Sigma) : u_- \in W^{\alpha,2}(\Omega_-),\ u_+ \in W^{1,2}(\Omega_+) \text{ and } u_- = \widetilde{w} u_+ \text{ on } \Sigma \Big\}.$$

5.3. Some open problems

We list a series of open questions that we hope will be of some interest to the reader. Besides other local-nonlocal transmission problems (as stated in the previous subsection), the following issues remain open for further study for our nonlocal transmission problem (1.8)–(1.12).

(a) A more refined regularity analysis to show the (Hölder) continuity of the solutions (up to the boundary) for our nonlocal-nonlocal transmission problem. We recall that a partial result is already available for the corresponding elliptic problem in Theorem 2.22 while the continuity up to the boundary is yet an open question for such systems. This issue is also closely related with the qualitative behavior of (u_+, u_-) near the interface Σ.

(b) A comprehensive analysis to develop sufficient conditions on the nonlinearities $f_\pm$ such that problem (1.8)–(1.12) is globally well-posed and study the further regularity of its corresponding solutions. The latter is essential to the study of the long-term asymptotic behavior of these systems, in terms of global attractors and ω-limit sets. In addition, the issue described in point (a) has further consequences on the dynamic behavior of solutions for (1.8)–(1.12). Indeed, the ω-limit sets of the problem (1.8)–(1.12) can exhibit a complicated structure if the function $f_\pm$ is non-monotone. In particular, this can happen if the stationary problem associated with (1.8)–(1.12) possess a continuum of nonconstant (steady-state) solutions. According to our discussion in [15] it may be possible to show that each globally defined regular solution $(u_+ (t), u_- (t))$ converges to a unique steady state $\left(u_+^*, u_-^*\right)$, as time goes to infinity, where $\left(u_+^*, u_-^*\right)$ is a proper solution of the corresponding stationary problem. A second issue is to investigate whether blow-up phenomenon occurs for problem (1.8)–(1.12), and whether it occurs in only one of the two regions (or both), or whether the linear term $\mathfrak{b}u_-$ present in the transmission condition (1.10) provides for a substantial dampening effect. Also, if blow-up occurs in finite time in some situations, where are the blow-up points? Can the blow-up occur near the interface Σ?

(c) In view of our recent work [14], other interesting transmission conditions can be considered for the parabolic problem (1.8) and (1.9), and a more general setting in which the interface Σ is rough (say, a d-dimensional fractal set with $N-1 < d < N$) may be developed. In particular when Σ is still a Lipschitz hypersurface, such transmission conditions may read

$$u_+ = wu_-, \quad \partial_t u_- + \mathcal{N}_w (u_+, u_-) + h (u_-) = 0, \quad \text{on } J \times \Sigma, \tag{5.8}$$

where h is allowed to be a nonlinear source/sink acting solely on the interfacial region Σ. In [14], a similar condition is used for a transmission problem consisting of a semilinear parabolic equation associated with the Laplacian Δ. A unified framework is developed for global existence of solutions, existence of finite dimensional attractors and blow-up phenomena for solutions under general conditions on the bulk and interfacial nonlinearities with (possible) competing behavior at infinity. Such questions remain open for the transmission problem (1.8) and (1.9), subject to conditions like (5.8), and suitable initial conditions for (u_+, u_-).

(d) In Remark 2.1, we have said that if using the fractional Laplacian (see (2.1)) in place of the regional fractional Laplacian, then it is not clear if this leads to a *local* transmission condition. This point should be investigated further. Indeed, if on the lower-side we have instead used the fractional Laplacian but kept the regional fractional Laplacian on the higher-order side (see Figure 1), it is still possible that we would get a local (albeit different) transmission condition. Then what if the fractional Laplacian was also used on the higher-order side, what do the transmission conditions look like? Is it still possible to develop a comparable framework for well-posedness and regularity, and then long-term behavior as well as blow-up?

We feel that these questions are worth investigating further by ourselves and/or the interested reader.

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References

[1] Bal, G., Ryzhik, L. (2000). Diffusion approximation of radiative transfer problems with interfaces. *SIAM J. Appl. Math.* 60:1887–1912.
[2] von Below, J., Nicaise, S. (1996). Dynamical interface transition in ramified media with diffusion. *Commun. Partial Differ. Equat.* 21:255–279.
[3] Bogdan, K., Burdzy, K., Chen, Z.-Q. (2003). Censored stable processes. *Probab. Theory Related Fields* 127:89–152.
[4] Borsuk, M. (2010). *Transmission Problems for Elliptic Second-Order Equations in Non-smooth Domains*. Frontiers in Mathematics, Basel: Birkhäuser, Springer.
[5] Chen, Z.-Q., Kumagai, T. (2003). Heat kernel estimates for stable-like processes on d-sets. *Stoch. Process. Appl.* 108:27–62.
[6] Danielli, D., Garofalo, N., Nhieu, D.-H. (2006). *Non-doubling Ahlfors Measures, Perimeter Measures, and the Characterization of the Trace Spaces of Sobolev Functions in Carnot–Carathéodory Spaces*. Mem. Am. Math. Soc., Vol. 182. Providence, RI: AMS.
[7] Davies, E.B. (1989). *Heat Kernels and Spectral Theory*. Cambridge: Cambridge University Press.
[8] Di Nezza, E., Palatucci, G., Valdinoci, E. (2012). Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* 136:521–573.
[9] Dipierro, S., Ros-Oton, X., Valdinoci, E. (to appear) Nonlocal problems with Neumann boundary conditions. *Rev. Mat. Iberoam.*
[10] Disser, K., Meyries, M., Rehberg, J. (2015). A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces. *J. Math. Anal. Appl.* 430:1102–1123.
[11] Dubkov, A.A., Spagnolo, B., Uchaikin, V.V. (2008). Lévy flight superdiffusion: An introduction. *Int. J. Bifurc. Chaos* 18:2649–2672.
[12] ter Elst, A.F.M., Meyries, M., Rehberg, J. (2014). Parabolic equations with dynamical boundary conditions and source terms on interfaces. *Ann. Mat. Pura Appl.* 193:1295–1318.
[13] Fukushima, M., Oshima, Y., Takeda, M. (2011). *Dirichlet Forms and Symmetric Markov Processes*. 2nd revised and extended edition. De Gruyter Studies in Mathematics, Vol. 19. Berlin: De Gruyter.
[14] Gal, C.G., Warma, M. (2016). Transmission problems with nonlocal boundary conditions and rough dynamic interfaces. *Nonlinearity* 29:161–197.

[15] Gal, C.G., Warma, M. (2016). Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions. *Discrete Contin. Dyn. Syst.* 36: 1279–1319.
[16] Gal, C.G., Warma, M. (2016). Long-term behavior of reaction-diffusion equations with nonlocal boundary conditions on rough domains. *Z. Angew. Math. Phys.* 67, Art. 83, 42 pp.
[17] Gal, C.G., Warma, M. (2016). Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions. *Evol. Equat. Control Theory* 5:61–103.
[18] Garroni, A., Müller, S. (2006). A variational model for dislocations in the line tension limit. *Arch. Ration. Mech. Anal.* 181:535–578.
[19] Gilkey, P., Kirsten, K. (2003). Heat content asymptotics with transmittal and transmission boundary conditions. *J. Lond. Math. Soc.* 68:431–443.
[20] Gorenflo, R., Mainardi, F., Vivoli, A. (2007). Continuous-time random walk and parametric subordination in fractional diffusion. *Chaos, Solitons Fractals* 34:87–103.
[21] Grisvard, P. (1985). *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, Vol. 24. Boston, MA: Pitman.
[22] Guan, Q.-Y. (2006). Integration by parts formula for regional fractional Laplacian. *Commun. Math. Phys.* 266:289–329.
[23] Guan, Q.-Y., Ma, Z.M. (2006). Reflected symmetric α-stable processes and regional fractional Laplacian. *Probab. Theory Related Fields* 134:649–694.
[24] Guan, Q.-Y., Ma, Z.M. (2005). Boundary problems for fractional Laplacians. *Stoch. Dyn.* 5:385–424.
[25] Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, Vol. 840. New York: Springer-Verlag.
[26] Huang, J., Zou, J. (2002). Some new a priori estimates for second-order elliptic and parabolic interface problems. *J. Differ. Equat.* 184:570–586.
[27] Humphries, N.E., Queiroz, N., Dyer, J.R., Pade, N.G., Musyl, M.K., Schaefer, K.M., Fuller, D.W., Brunnschweiler, J.M., Doyle, T.K., Houghton, J.D., Hays, G.C., Jones, C.S., Noble, L.R., Wearmouth, V.J., Southall, E.J., Sims, D.W. (2010). Environmental context explains Lévy and Brownian movement patterns of marine predators. *Nature* 465:1066–1069.
[28] Hung, P.H., Sanchez-Palencia, E. (1974). Phenomènes de transmission à travers des couches minces de conductivité élevée (Phenomena of transmissions through thin layers of high conductivity). *J. Math. Anal. Appl.* 47:284–309.
[29] Jara, M. (2009). Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps. *Commun. Pure Appl. Math.* 62:198–214.
[30] Jonsson, A., Wallin, H. (1984). Function spaces on subsets of $\mathbb{R}^N$. *Math. Rep.* 2.
[31] Kriventsov, D. (2015). Regularity for a local-nonlocal transmission problem. *Arch. Ration. Mech. Anal.* 217:1103–1195.
[32] Kriventsov, D. (2015). A local-nonlocal transmission problem. University of Texas Electronic Theses and Dissertations. Available at: http://hdl.handle.net/2152/31513 (accessed 2 October 2015).
[33] Lancia, M.R. (2002). A transmission problem with a fractal interface. *Z. Anal. Anwendungen* 21:113–133.
[34] Lancia, M.R. (2003). Second order transmission problems across a fractal surface. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* 5:191–213.
[35] Lancia, M.R., Vernole, P. (2012). Semilinear evolution transmission problems across fractal layers. *Nonlinear Anal.* 75:4222–4240.
[36] Lancia, M.R., Vernole, P. (2010). Irregular heat flow problems. *SIAM J. Math. Anal.* 42:1539–1567.
[37] Lions, J.-L., Magenes, E. (1972). *Non-homogeneous Boundary Value Problems and Applications*, Vol. I. New York-Heidelberg: Springer-Verlag.
[38] Mandelbrot, B.B., Van Ness, J. W. (1969). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* 10:422–437.
[39] Maz'ya, V.G. (1985). *Sobolev Spaces*. Berlin: Springer-Verlag.
[40] Mellet, A., Mischler, S., Mouhot, C. (2011). Fractional diffusion limit for collisional kinetic equations. *Arch. Ration. Mech. Anal.* 199:493–525.

[41] Mou, C., Yi, Y. (2015). Interior regularity for regional fractional Laplacian. *Commun. Math. Phys.* 340:233–251.
[42] Nomirovskii, D. (2007). Generalized solvability and optimization of a parabolic system with a discontinuous solution. *J. Differ. Equat.* 233:1–21.
[43] Ouhabaz, E.M. (2005). *Analysis of Heat Equations on Domains*. London Mathematical Society Monographs Series, Vol. 31. Princeton, NJ: Princeton University Press.
[44] Rothe, F. (1984). *Global Solutions of Reaction-Diffusion Systems*. Lecture Notes in Math., Vol. 1072. Berlin, New York: Springer-Verlag.
[45] Reynolds, A.M., Rhodes, C.J. (2009). The Lévy flight paradigm: Random search patterns and mechanism. *Ecology* 90:877–887.
[46] Schertzer, D., Larcheveque, M., Duan, J., Yanovsky, V.V., Lovejoy, S. (2001). Fractional Fokker-Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises. *J. Math. Phys.* 42:200–212.
[47] Schneider, W.R. (1990). Grey noise. In: Albeverio, S., Casati, G. Cattaneo, U., Merlini, D., Moresi, R., eds. *Stochastic Processes, Physics and Geometry*. Teaneck, NJ, USA: World Scientific, pp. 676–681.
[48] Teresi, L., Vacca, E. (2007). *Transmission Phenomena Across Highly Conductive Interfaces*. Applied and Industrial Mathematics in Italy II. Ser. Adv. Math. Appl. Sci., Vol. 75. Hackensack, NJ: World Sci. Publ., pp. 585–596.
[49] Valdinoci, E. (2009). From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SeMA* 49:33–44.
[50] Warma, M. (2015). The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets. *Potential Anal.* 42:499–547.
[51] Warma, M. (2016). The fractional Neumann and Robin type boundary conditions for the regional fractional p-Laplacian. *NoDEA Nonlinear Differ. Equat. Appl.* 23, Art. 1, 46 pp.
[52] Zolesio, J.L. (1977). Multiplication dans les espaces de Besov. *Proc. Royal Soc. Edinburgh (A)* 78:113–117.

Appendix: Some technical tools

We first state a result that gives an estimate on time convolution integrals involving the bounded operator $S_{\alpha,\beta}(t)$.

Lemma A. *Define* $\mathfrak{n} := N/(2\beta)$ *for* $N \geq 2$ *and* $\beta \in (0, 1)$. *Let* $p, s_1 \in [1, \infty]$, $s_2 \in (1, \infty]$, $\theta, \varepsilon \in [0, 1]$, $\delta \in [0, \infty)$ *satisfy*

$$\frac{\mathfrak{n}}{s_1} + \frac{1}{s_2} < 1 + \frac{\mathfrak{n}}{p}, \quad \frac{\mathfrak{n}}{s_1} + \frac{1}{s_2} + \theta + \varepsilon \leq 1 + \frac{\mathfrak{n}}{p} + \delta$$

and

$$\frac{1}{s_2} + \theta < 1, \quad \frac{1}{s_2} + \theta + \varepsilon \leq 1 + \delta.$$

Let $f : [0, \infty) \to \mathbb{R}$ *be a measurable function such that*

$$p_{s_2}(f) := \sup_{t_1,t_2 \in [0,\infty), 0 \leq t_2 - t_1 \leq 1} \left(\int_{t_1}^{t_2} |f(t)|^{s_2} \, dt \right)^{\frac{1}{s_2}} < \infty.$$

Define the function

$$g(t) := (t \wedge 1)^{\delta} \int_0^t \left\| S_{\alpha,\beta}(t - \tau) \right\|_{p,s_1} (\tau \wedge 1)^{-\theta} f(\tau) \, d\tau, \ \forall\, t \geq 0.$$

Then there exists a constant $C > 0$ independent of t such that

$$\left|g(t)\right| \leq C(t \wedge 1)^{\varepsilon} p_{s_2}(f), \text{ for all } t \in (0, \infty).$$

Proof. The claim follows from combining the proof of [44, Lemma 6] with the ultracontractivity estimates of Theorem 2.16 (see, in particular, (2.38)). □

The ultracontractivity properties of the semigroup $S_{\alpha,\beta}$ (see (2.38)) allow us to also deduce from the proof of [44, Lemma 4] the following lemma.

Lemma B. *Let $p, q \in [1, \infty]$ such that $p < q$. Given a subset $\Pi \subset X^p(\Omega\backslash\Sigma)$, assume that*

$$\kappa(\Pi) := \left\{\frac{u}{\|u\|_{L^p(\Omega\backslash\Sigma)}} : u \in \Pi,\ u \neq 0\right\}$$

is precompact in $X^p(\Omega\backslash\Sigma)$. Here by $u \neq 0$ on $\Omega\backslash\Sigma$, we mean that $u_+ \neq 0$ (a.e.) on Ω_+ and $u_- \neq 0$ (a.e.) on Ω_-. Then there exists a nondecreasing function $g : [0, \infty) \to [0, 1]$, depending only on p, q, α, β and Π such that:

(a) *For all $t > 0$ and $u \in \Pi$,*

$$\left\|S_{\alpha,\beta}(t)u\right\|_{L^q(\Omega\backslash\Sigma)} \leq Cg(t)(t \wedge 1)^{-\mathfrak{n}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p(\Omega\backslash\Sigma)}. \tag{A.1}$$

(b) *We have $\lim_{t\to 0^+} g(t) = 0$. The function $W = W(t)$ defined by*

$$(W(t))^{-\mathfrak{n}\left(\frac{1}{p}-\frac{1}{q}\right)} = g(t)(t \wedge 1)^{-\mathfrak{n}\left(\frac{1}{p}-\frac{1}{q}\right)}$$

has the properties

$$\lim_{t\to 0^+} W(t) = 0 \text{ and } (t \wedge 1) \leq W(t) \leq (t \wedge 1)^{\frac{1}{2}}.$$

Lemma C. *Consider the following cases:*

(a) *Let $p_0, \gamma \in [1, \infty)$ satisfy $(\gamma - 1)\mathfrak{n}/p_0 < 1$.*

(b) *Let $p_0, \gamma \in (1, \infty)$ satisfy $(\gamma - 1)\mathfrak{n}/p_0 = 1$.*

Then there exist $\varepsilon \in (0, 1)$, $k \in \mathbb{N}$ and finite sequences $\{p_i\}$, $\{\delta_i\}$ such that $\delta_i \in (0, 1)$ and $p_0 < p_1 < \cdots < p_k = \infty$, for $i = 1, \ldots, k$. In addition, the following are satisfied:

$$\begin{cases} (\gamma - 1)\left(\delta_i + \dfrac{\mathfrak{n}}{p_i}\right) + \varepsilon < 1, & \text{for } i = 1, \ldots, k;\ i \neq 1 \text{ in case (b).} \\ \dfrac{\gamma}{p_i} \leq 1, & \text{for } i = 1, \ldots, k. \\ \gamma\delta_i < 1, & \text{for } i = 1, \ldots, k. \\ \dfrac{\mathfrak{n}}{p_{i-1}} - \dfrac{\mathfrak{n}}{p_i} = \delta_i, & \text{for } i = 1, \ldots, k. \end{cases}$$

Proof. Apply the assertion of [44, Lemma 12] with $q_1 = q_2 = \infty$. □

Lemma D. *Let $p_0 \in [1,\infty]$ be arbitrary and $r_1, r_2 \in (0,\infty]$, $\gamma \in [1,\infty)$, $b \in [0,1]$ such that*

$$\gamma\,(1-b)\left(\frac{\mathfrak{n}}{r_1}+\frac{1}{r_2}\right)<1,\quad \gamma\,\frac{1-b}{r_1}\le 1 \text{ and } \gamma b\le 1.$$

Then there exist $\varepsilon \in (0,1)$, $k \in \mathbb{N}$ and finite sequences $\{p_i\}$, $\{\delta_i\}$ such that $\delta_i \in (0,1)$ and $p_0 < p_1 < \cdots < p_k = \infty$, for $i = 1,\ldots,k$. In addition, the following are satisfied:

$$\begin{cases} \gamma\dfrac{1-b}{r_1}+\gamma\dfrac{b}{p_i}\le 1, & \text{for } i=1,\ldots,k;\\[2mm] \gamma\dfrac{1-b}{r_2}+\gamma b\delta_i<1-\varepsilon, & \text{for } i=1,\ldots,k.\\[2mm] \delta_i=\dfrac{\mathfrak{n}}{p_{i-1}}-\dfrac{\mathfrak{n}}{p_i}, & \text{for } i=1,\ldots,k.\end{cases}$$

Proof. Apply the assertion of [44, Lemma 16] with $q_1 = q_2 = \infty$. □

The following basic "feedback" inequality is taken from [44, Lemma 18].

Lemma E. *Let $y, z_0, z_1 \in [0,\infty)$ and $\sigma \in (0,1)$ be such that $y \le z_0 + z_1 y^{\sigma}$. Then*

$$y\le \frac{z_0}{1-\sigma}+z_1^{\frac{1}{1-\sigma}}.$$

The following result on pointwise multiplication of functions in Sobolev spaces can be used to ensure minimal (optimal) regularity on the weight w. Assume that Σ is smooth enough (at least Lipschitz continuous) and $s_1, s_2, s \ge 0$, $N \ge 1$.

Lemma F (see [52]). *Let s, s_1, and s_2 be real numbers satisfying*

$$min(s_1,s_2)\ge s \qquad \text{and} \qquad s_1+s_2-s>\frac{N-1}{2},$$

where the strictness of the last two inequalities can be interchanged if $s \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Then, for any $w_1 \in W^{s_1,2}(\Sigma)$, $w_2 \in W^{s_2,2}(\Sigma)$ the product $w_1 w_2 \in W^{s,2}(\Sigma)$ and there exists a constant $C = C(N, s_1, s_2, s) > 0$, independent of w_1, w_2 such that

$$\|w_1w_2\|_{W^{s,2}(\Sigma)}\le C\,\|w_1\|_{W^{s_1,2}(\Sigma)}\,\|w_2\|_{W^{s_2,2}(\Sigma)}\,.$$